Was von Neumann Right After All?

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Abstract

The work with TGD inspired model for quantum computation led to the realization that von Neumann algebras, in particular hyper-finite factors of type II_1 could provide the mathematics needed to develop a more explicit view about the construction of S-matrix. This has turned out to be the case to the extend that a general master formula for S-matrix with interactions described as a deformation of ordinary tensor product to Connes tensor products emerges. The theory leads also to a prediction for the spectrum of Planck constants associated with M^4 and CP_2 degrees of freedom.

1. Some background

It has been for few years clear that TGD could emerge from the mere infinite-dimensionality of the Clifford algebra of infinite-dimensional "world of classical worlds" and from number theoretical vision in which classical number fields play a key role and determine imbedding space and space-time dimensions. This would fix completely the "world of classical worlds".

Infinite-dimensional Clifford algebra is a standard representation for von Neumann algebra known as a hyper-finite factor of type II_1. In TGD framework the infinite tensor power of C(8), Clifford algebra of 8-D space would be the natural representation of this algebra.

2. How to localize infinite-dimensional Clifford algebra?

The basic new idea is to make this algebra *local*: local Clifford algebra as a generalization of gamma field of string models.

a) Represent Minkowski coordinate of M^d as linear combination of gamma matrices of D-dimensional space. This is the first guess. One fascinating finding is that this notion can be quantized and classical M^d is genuine quantum M^d with coordinate values eigenvalues of quantal commuting Hermitian operators built from matrix elements. Euclidian space is not obtained in this manner. Minkowski signature is something quantal and the standard quantum group $Gl_(2,q)(C)$ with (non-Hermitian matrix elements) gives M^4.

b) Form power series of the M^d coordinate represented as linear combination of gamma matrices with coefficients in corresponding infinite-D Clifford algebra. You would get tensor product of two algebra.

c) There is however a problem: one cannot distinguish the tensor product from the original infinite-D Clifford algebra. $D = 8$ is however an exception! You can replace gammas in the expansion of M^8 coordinate by hyper-octonionic units which are non-associative (or octonionic units in quantum complexified-octonionic case). Now you cannot anymore absorb the tensor factor to the Clifford algebra and you get genuine M^8-localized factor of type II_1. Everything is determined by infinite-dimensional gamma matrix fields analogous to conformal super fields with z replaced by hyperoctonion.

d) Octonionic non-associativity actually reproduces whole classical and quantum TGD: space-time surface must be associative sub-manifolds hence hyper-quaternionic surfaces of M^8. Representability as surfaces in $M^4 \times CP_2$ follows naturally, the notion of configuration space of 3-surfaces, etc....

3. Connes tensor product for free fields as a universal definition of interaction quantum field theory

This picture has profound implications. Consider first the construction of S-matrix.

a) A non-perturbative construction of S-matrix emerges. The deep principle is simple. The canonical outer automorphism for von Neumann algebras defines a natural candidate unitary transformation giving rise to propagator. This outer automorphism is trivial for II_1 factors meaning that all lines appearing in Feynman diagrams must be on mass shell states satisfying Super Virasoro conditions. You can allow all possible diagrams: all on mass shell loop corrections vanish by unitarity and what remains are diagrams with single N-vertex.

b) At 2-surface representing N-vertex space-time sheets representing generalized Bohr orbits of incoming and outgoing particles meet. This vertex involves von Neumann trace (finite!) of localized gamma matrices expressible in terms of fermionic oscillator operators and defining free fields satisfying Super Virasoro conditions.

c) For free fields ordinary tensor product would not give interacting theory. What makes S-matrix non-trivial is that *Connes tensor product* is used instead of the ordinary one. This tensor product is a universal description for interactions and we can forget perturbation theory! Interactions result as a deformation of tensor product. Unitarity of resulting S-matrix is unproven but I dare believe that it holds true.

d) The subfactor $\mathcal{N}$ defining the Connes tensor product has interpretation in terms of the interaction between experimenter and measured system and each interaction type defines its own Connes tensor product. Basically $\mathcal{N}$ represents the limitations of the experimenter. For instance, IR and UV cutoffs could be seen as primitive manners to describe what $\mathcal{N}$ describes much more elegantly. At the limit when $\mathcal{N}$ contains only single element, theory would become free field theory but this is ideal situation never achievable.

e) Large $\hbar$ phases provide good hopes of realizing topological quantum computation. There is an additional new element. For quantum spinors state function reduction cannot be performed unless quantum deformation parameter equals to $q = 1$. The reason is that the components of quantum spinor do not commute: it is however possible to measure the commuting operators representing moduli squared of the components giving the probabilities associated with 'true' and 'false'. The universal eigenvalue spectrum for probabilities does not in general contain (1,0) so that quantum qbits are inherently fuzzy. State function reduction would occur only after a transition to q=1 phase and decoherence is not a problem as long as it does not induce this transition.

1 Introduction

The work with TGD inspired model [E9] for topological quantum computation [70] led to the realization that von Neumann algebras [16, 17, 18, 19], in particular so called hyper-finite factors of type II_1 [21], seem to provide the mathematics needed to develop a more explicit view about the construction of S-matrix. I have already discussed a vision for how to achieve this [C7]. In this chapter I will discuss various aspects of type II_1 factors and their physical interpretation in TGD framework. The lecture notes of R. Longo [20] give a concise and readable summary about the basic definitions and results related to von Neumann algebras and I have used this material freely in this chapter.

1.1 Philosophical ideas behind von Neumann algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation * and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator A belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type II_1 [21].

The definitions of adopted by von Neumann allow however more general algebras. Type I_n algebras correspond to finite-dimensional matrix algebras with finite traces whereas I_∞ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type III non-trivial traces are always infinite and the notion of trace becomes useless.

1.2 Von Neumann, Dirac, and Feynman

The association of algebras of type I with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type II_1 as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac [22] based on the notion of delta function, plus the emergence of s [25], the possibility to formulate the notion of delta function rigorously in terms of distributions [23, 24], and the emergence of path integral approach [26] meant that von Neumann approach was forgotten by particle physicists.

Algebras of type II_1 have emerged only much later in conformal and topological quantum field theories [30, 31] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [28, 29] relate closely to type II_1 factors. In topological quantum computation [70] based on braid groups [27] modular S-matrices they play an especially important role.

In algebraic quantum field theory [34] defined in Minkowski space the algebras of observables associated with bounded space-time regions correspond quite generally to the type III_1 hyper-finite factor [35, 36].

1.3 Factors of type II_1 and quantum TGD

For me personally the realization that TGD Universe is tailored for topological quantum computation [E9] led also to the realization that hyper-finite (ideal for numerical approximations) von Neumann algebras of type II_1 have a direct relevance for TGD.

The basic facts about hyper-finite von Neumann factors of type II_1 suggest a more concrete view about the general mathematical framework needed.

1. The effective 2-dimensionality of the construction of quantum states and configuration space geometry in quantum TGD framework makes hyper-finite factors of type II_1 very natural as operator algebras of the state space. Indeed, the generators of conformal algebras, the gamma matrices of the configuration space, and the modes of the induced spinor fields are labelled by discrete labels. Hence the tangent space of the configuration space is a separable Hilbert space and its Clifford algebra is a hyper-finite type II_1 factor. Super-symmetry requires that the bosonic algebra generated by configuration space Hamiltonians and the Clifford algebra of configuration space both correspond to hyper-finite type II_1 factors.

2. Four-momenta relate to the positions of tips of future and past directed light cones appearing naturally in the construction of S-matrix. In fact, configuration space of 3-surfaces can be regarded as union of big-bang/big crunch type configuration spaces obtained as a union of light-cones parameterized by the positions of their tips. The algebras of observables associated with bounded regions of M^4 are hyper-finite and of type III_1 in algebraic quantum

field theory [35]. The algebras of observables in the space spanned by the tips of these light-cones are not needed in the construction of S-matrix so that there are good hopes of avoiding infinities coming from infinite traces.

3. Many-sheeted space-time concept forces to refine the notion of sub-system. Jones inclusions $\mathcal{N} \subset \mathcal{M}$ for factors of type II_1 define in a generic manner to imbed interacting sub-systems to a universal II_1 factor which now naturally corresponds to the infinite Clifford algebra of the tangent space of configuration space of 3-surfaces and contains interaction as $\mathcal{M} : \mathcal{N}$-dimensional analog of tensor factor. Topological condensation of space-time sheet to a larger space-time sheet, the formation of bound states by the generation of join along boundaries bonds, interaction vertices in which space-time surface branches like a line of Feynman diagram: all these situations might be described by Jones inclusion [40, 41] characterized by the Jones index $\mathcal{M} : \mathcal{N}$ assigning to the inclusion also a minimal conformal field theory and quantum group in case of $\mathcal{M} : \mathcal{N} < 4$ and conformal theory with $k = 1$ Kac Moody for $\mathcal{M} : \mathcal{N} = 4$ [39].

4. Von Neumann's somewhat artificial idea about identical a priori probabilities for states could replaced with the finiteness requirement of quantum theory. Indeed, it is traces which produce the infinities of quantum field theories. That $\mathcal{M} : \mathcal{N} = 4$ option is not realized physically as quantum field theory (it would rather correspond to string model type theory characterized by a Kac-Moody algebra instead of quantum group), could correspond to the fact that dimensional regularization works only in $D = 4 - \epsilon$. Dimensional regularization with space-time dimension $D = 4 - \epsilon \rightarrow 4$ could be interpreted as the limit $\mathcal{M} : \mathcal{N} \rightarrow 4$. $\mathcal{M}$ as an $\mathcal{M} : \mathcal{N}$-dimensional $\mathcal{N}$-module would provide a concrete model for a quantum space with non-integral dimension as well as its Clifford algebra. An entire sequence of regularized theories corresponding to the allowed values of $\mathcal{M} : \mathcal{N}$ would be predicted.

1.4 How to localize infinite-dimensional Clifford algebra?

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1.5 Non-trivial S-matrix from the Connes tensor product for free fields

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1.5.2 Equivalence of generalized loop diagrams with tree diagrams or vanishing of loop corrections or both?

The work with bi-algebras [C7] led to the proposal that the generalized Feynman diagrams of TGD at space-time level satisfy a generalization of the duality of old-fashioned string models.

Generalized Feynman diagrams containing loops are equivalent with tree diagrams so that they could be interpreted as representing computations or analytic continuations. This symmetry can be formulated as a condition on algebraic structures generalizing bi-algebras. The new element are the vacuum lines having a natural counterpart at the level of bi-algebras and braid diagrams. At space-time level they correspond to vacuum extremals.

An alternative for this symmetry is the vanishing of loop corrections. The first symmetry seems to be natural for braiding S-matrices associated with external lines of generalized Feynman diagrams and the latter for the entire S-matrix.

1.6 Cognitive consciousness, quantum computations, and Jones inclusions

Large $\hbar$ phases provide good hopes of realizing topological quantum computation. There is an additional new element. For quantum spinors state function reduction cannot be performed unless quantum deformation parameter equals to $q = 1$. The reason is that the components of quantum spinor do not commute: it is however possible to measure the commuting operators representing moduli squared of the components giving the probabilities associated with 'true' and 'false'. The universal eigenvalue spectrum for probabilities does not in general contain (1,0) so that quantum qbits are inherently fuzzy. State function reduction would occur only after a transition to q=1 phase and decoherence is not a problem as long as it does not induce this transition.

2 Von Neumann algebras

In this section basic facts about von Neumann algebras are summarized using as a background material the concise summary given in the lecture notes of Longo [20].

2.1 Basic definitions

A formal definition of von Neumann algebra [17, 18, 19] is as a *-subalgebra of the set of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ closed under weak operator topology, stable under the conjugation $J =^*$: $x \rightarrow x^*$, and containing identity operator Id. This definition allows also von Neumann algebras for which the trace of the unit operator is not finite.

Identity operator is the only operator commuting with a simple von Neumann algebra. A general von Neumann algebra allows a decomposition as a direct integral of simple algebras, which von Neumann called factors. Classification of von Neumann algebras reduces to that for factors.

$\mathcal{B}(\mathcal{H})$ has involution * and is thus a *-algebra. $\mathcal{B}(\mathcal{H})$ has order order structure $A \geq 0 : (Ax, x) \geq 0$. This is equivalent to $A = BB^*$ so that order structure is determined by algebraic structure. $\mathcal{B}(\mathcal{H})$ has metric structure in the sense that norm defined as supremum of $||Ax||$, $||x|| \leq 1$ defines the notion of continuity. $||A||^2 = inf\{\lambda > 0 : AA^* \leq \lambda I\}$ so that algebraic structure determines metric structure.

There are also other topologies for $\mathcal{B}(\mathcal{H})$ besides norm topology.

1. $A_i \rightarrow A$ strongly if $||Ax - A_ix|| \rightarrow 0$ for all x. This topology defines the topology of C^* algebra. $\mathcal{B}(\mathcal{H})$ is a Banach algebra that is $||AB|| \leq ||A|| \times ||B||$ (inner product is not necessary) and also C^* algebra that is $||AA^*|| = ||A||^2$.

2. $A_i \rightarrow A$ weakly if $(A_ix, y) \rightarrow (Ax, y)$ for all pairs (x, y) (inner product is necessary). This topology defines the topology of von Neumann algebra as a sub-algebra of $\mathcal{B}(\mathcal{H})$.

Denote by M' the commutant of $\mathcal{M}$ which is also algebra. Von Neumann's bicommutant theorem says that $\mathcal{M}$ equals to its own bi-commutant. Depending on whether the identity operator has a finite trace or not, one distinguishes between algebras of type II_1 and type II_∞. II_1 factor allow trace with properties $tr(Id) = 1$, $tr(xy) = tr(yx)$, and $tr(x^*x) > 0$, for all $x \neq 0$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained by completing $\mathcal{M}$ respect to the inner product defined $\langle x|y\rangle = tr(x^*y)$ defines inner product in $\mathcal{M}$ interpreted as Hilbert space. The normalized trace induces a trace in M', natural trace $Tr_{M'}$, which is however not necessarily normalized. JxJ defines an element of M': if $\mathcal{H} = L^2(\mathcal{M})$, the natural trace is given by $Tr_{M'}(JxJ) = tr_M(x)$ for all $x \in M$ and bounded.

2.2 Basic classification of von Neumann algebras

Consider first some definitions. First of all, Hermitian operators with positive trace expressible as products xx^* are of special interest since their sums with positive coefficients are also positive.

In quantum mechanics Hermitian operators can be expressed in terms of projectors to the eigen states. There is a natural partial order in the set of isomorphism classes of projectors by inclusion: $E < F$ if the image of $\mathcal{H}$ by E is contained to the image of $\mathcal{H}$ by a suitable isomorph of F. Projectors are said to be metrically equivalent if there exist a partial isometry which maps the images $\mathcal{H}$ by them to each other. In the finite-dimensional case metric equivalence means that isomorphism classes are identical $E = F$.

The algebras possessing a minimal projection E_0 satisfying $E_0 \leq F$ for any F are called type I algebras. Bounded operators of n-dimensional Hilbert space define algebras I_n whereas the bounded operators of infinite-dimensional separable Hilbert space define the algebra I_∞. I_n and I_∞ correspond to the operator algebras of quantum mechanics. The states of harmonic oscillator correspond to a factor of type I.

The projection F is said to be finite if $F < E$ and $F \equiv E$ implies $F = E$. Hence metric equivalence means identity. Simple von Neumann algebras possessing finite projections but no minimal projections so that any projection E can be further decomposed as $E = F + G$, are called factors of type II.

Hyper-finiteness means that any finite set of elements can be approximated arbitrary well with the elements of a finite-dimensional sub-algebra. The hyper-finite II_∞ algebra can be regarded as a tensor product of hyper-finite II_1 and I_∞ algebras. Hyper-finite II_1 algebra can be regarded as a Clifford algebra of an infinite-dimensional separable Hilbert space sub-algebra of I_∞.

Hyper-finite II_1 algebra can be constructed using Clifford algebras $C(2n)$ of $2n$-dimensional spaces and identifying the element x of $2^n \times 2^n$ dimensional $C(n)$ as the element $diag(x,x)/2$ of $2^{n+1} \times 2^{n+1}$-dimensional $C(n+1)$. The union of algebras $C(n)$ is formed and completed in the weak operator topology to give a hyper-finite II_1 factor. This algebra defines the Clifford algebra of infinite-dimensional separable Hilbert space and is thus a sub-algebra of I_∞ so that hyper-finite II_1 algebra is more regular than I_∞.

von Neumann algebras possessing no finite projections (all traces are infinite or zero) are called algebras of type III. It was later shown by Connes [37] that these algebras are labelled by a parameter varying in the range $[0,1]$, and referred to as algebras of type III_x. III_1 category contains a unique hyper-finite algebra. It has been found that the algebras of observables associated with bounded regions of 4-dimensional Minkowski space in quantum field theories correspond to hyper-finite factors of type III_1 [20]. Also statistical systems at finite temperature correspond to factors of type III and temperature parameterizes one-parameter set of automorphisms of this algebra [35]. Zero temperature limit correspond to I_∞ factor and infinite temperature limit to II_1 factor.

2.3 Non-commutative measure theory and non-commutative topologies and geometries

von Neumann algebras and C^* algebras give rise to non-commutative generalizations of ordinary measure theory (integration), topology, and geometry. It must be emphasized that these structures are completely natural aspects of quantum theory. In particular, for the hyper-finite type II_1 factors quantum groups and Kac Moody algebras [39] emerge quite naturally without any need for ad hoc modifications such as making space-time coordinates non-commutative. The effective 2-dimensionality of quantum TGD (partonic or stringy 2-surfaces code for states) means that these structures appear completely naturally in TGD framework.

2.3.1 Non-commutative measure theory

von Neumann algebras define what might be a non-commutative generalization of measure theory and probability theory [20].

1. Consider first the commutative case. Measure theory is something more general than topology since the existence of measure (integral) does not necessitate topology. Any measurable function f in the space $L^\infty(X,\mu)$ in measure space (X,μ) defines a bounded operator M_f in the space $\mathcal{B}(L^2(X,\mu))$ of bounded operators in the space $L^2(X,\mu)$ of square integrable functions with action of M_f defined as $M_f g = fg$.

2. Integral over $\mathcal{M}$ is very much like trace of an operator $f_{x,y} = f(x)\delta(x,y)$. Thus trace is a natural non-commutative generalization of integral (measure) to the non-commutative case and defined for von Neumann algebras. In particular, generalization of probability measure results if the case $tr(Id) = 1$ and algebras of type I_n and II_1 are thus very natural from the point of view of non-commutative probability theory.

The trace can be expressed in terms of a cyclic vector Ω or vacuum/ground state in physicist's terminology. Ω is said to be cyclic if the completion $\overline{M\Omega} = H$ and separating if $x\Omega$ vanishes only for $x = 0$. Ω is cyclic for $\mathcal{M}$ if and only if it is separating for M'. The expression for the trace given by

$$Tr(ab) = \left(\frac{(ab+ba)}{2}, \Omega\right) \quad (1)$$

is symmetric and allows to defined also inner product as $(a,b) = Tr(a^*b)$ in $\mathcal{M}$. If Ω has unit norm $(\Omega,\Omega) = 1$, unit operator has unit norm and the algebra is of type II_1. Fermionic oscillator operator algebra with discrete index labelling the oscillators defines II_1 factor. Group algebra is second example of II_1 factor.

The notion of probability measure can be abstracted using the notion of state. State ω on a C^* algebra with unit is a positive linear functional on $\mathcal{U}$, $\omega(1) = 1$. By so called KMS construction [20] any state ω in C^* algebra $\mathcal{U}$ can be expressed as $\omega(x) = (\pi(x)\Omega,\Omega)$ for some cyclic vector Ω and π is a homomorphism $\mathcal{U} \to \mathcal{B}(\mathcal{H})$.

2.3.2 Non-commutative topology and geometry

C^* algebras generalize in a well-defined sense ordinary topology to non-commutative topology.

1. In the Abelian case Gelfand Naimark theorem [20] states that there exists a contravariant functor F from the category of unital abelian C^* algebras and category of compact topological spaces. The inverse of this functor assigns to space X the continuous functions f on X

with norm defined by the maximum of f. The functor assigns to these functions having interpretation as eigen states of mutually commuting observables defined by the function algebra. These eigen states are delta functions localized at single point of X. The points of X label the eigenfunctions and thus define the spectrum and obviously span X. The connection with topology comes from the fact that continuous map $Y \rightarrow X$ corresponds to homomorphism $C(X) \rightarrow C(Y)$.

2. In non-commutative topology the function algebra $C(X)$ is replaced with a general C^* algebra. Spectrum is identified as labels of simultaneous eigen states of the Cartan algebra of C^* and defines what can be observed about non-commutative space X.

3. Non-commutative geometry can be very roughly said to correspond to *-subalgebras of C^* algebras plus additional structure such as symmetries. The non-commutative geometry of Connes [38] is a basic example here.

2.4 Modular automorphisms

von Neumann algebras allow a canonical unitary evolution associated with any state ω fixed by the selection of the vacuum state Ω [20]. This unitary evolution is an automorphism fixed apart form unitary automorphisms $A \rightarrow UAU^*$ related with the choice of Ω.

Let ω be a normal faithful state: $\omega(x^*x) > 0$ for any x. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \rightarrow x\Omega$. The conjugation * in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \rightarrow x^*\Omega$. The closure of S_0 is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. Δ is positive self-adjoint operator and J anti-unitary involution. The following conditions are satisfied

$$
\begin{aligned}
\Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\
J\mathcal{M}J &= \mathcal{M}' \ .
\end{aligned}
\tag{2}
$$

Δ^{it} is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation ω as $\pi \rightarrow \Delta^{it}\pi\Delta^{-it}$.

2.5 Joint modular structure and sectors

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion. The unitary operator $\gamma = J_N J_M$ defines a canonical endomorphisms $M \rightarrow N$ in the sense that it depends only up to inner automorphism on $\mathcal{N}$, γ defines a sector of $\mathcal{M}$. The sectors of $\mathcal{M}$ are defined as $Sect(\mathcal{M}) = End(\mathcal{M})/Inn(\mathcal{M})$ and form a semi-ring with respected to direct sum and composition by the usual operator product. It allows also conjugation.

$L^2(\mathcal{M})$ is a normal bi-module in the sense that it allows commuting left and right multiplications. For $a, b \in M$ and $x \in L^2(\mathcal{M})$ these multiplications are defined as $axb = aJb^*Jx$ and it is easy to verify the commutativity using the factor $Jy^*J \in \mathcal{M}'$. Connes [38] has shown that all normal bi-modules arise in this way up to unitary equivalence so that representation concepts make sense. It is possible to assign to any endomorphism ρ index $Ind(\rho) \equiv M : \rho(\mathcal{M})$. This means that the sectors are in 1-1 correspondence with inclusions. For instance, in the case of hyper-finite II_1 they are labelled by Jones index. Furthermore, the objects with non-integral dimension $\sqrt{[\mathcal{M} : \rho(\mathcal{M})]}$ can be identified as quantum groups, loop groups, infinite-dimensional Lie algebras, etc...

3 Inclusions of II_1 and III_1 factors

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. For type I algebras the inclusions are trivial and tensor product description applies as such. For factors of II_1 and III the inclusions are highly non-trivial. The inclusion of type II_1 factors were understood by Vaughan Jones [40]and those of factors of type III by Alain Connes [37].

Sub-factor $\mathcal{N}$ of $\mathcal{M}$ is defined as a closed *-stable C-subalgebra of $\mathcal{M}$. Let $\mathcal{N}$ be a sub-factor of type II_1 factor $\mathcal{M}$. Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = dim_N(L^2(\mathcal{M})) = Tr_{N'}(id_{L^2(\mathcal{M})})$. One can say that the dimension of completion of $\mathcal{M}$ as $\mathcal{N}$ module is in question.

3.1 Basic findings about inclusions

What makes the inclusions non-trivial is that the position of $\mathcal{N}$ in $\mathcal{M}$ matters. This position is characterized in case of hyper-finite II_1 factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of $\mathcal{M}$ as $\mathcal{N}$ module and also as the inverse of the dimension defined by the trace of the projector from $\mathcal{M}$ to $\mathcal{N}$. It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either $\mathcal{M}$ or $\mathcal{M}$, only the imbedding.

The basic facts proved by Jones are following [40].

1. For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$\begin{aligned} &a) \;\; \mathcal{M} : \mathcal{N} = 4cos^2(\pi/h) \;\; , \quad h \geq 3 \;\; , \\ &b) \;\; \mathcal{M} : \mathcal{N} \geq 4 \;\; . \end{aligned} \qquad (3)$$

   the numbers at right hand side are known as Beraha numbers [32]. The comments below give a rough idea about what finiteness of principal graph means.

2. As explained in [39], for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (dimg(g) - r)/r$. The Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of SU(2) and the interpretation proposed in [45] is following. The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: SU(2) itself, circle group U(1), and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection. The diagrams corresponding to finite subgroups are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with n=6,7,8 for tedrahedron, cube, dodecahedron. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed.

The interpretation of [45] is that the subfactors correspond to inclusions $\mathcal{N} \subset \mathcal{M}$ defined in the following manner.

1. Let G be a finite subgroup of SU(2). Denote by R the infinite-dimensional Clifford algebras resulting from infinite-dimensional tensor power of $M_2(C)$ and by R_0 its subalgebra obtained by restricting $M_2(C)$ element of the first factor to be unit matrix. Let G act by automorphisms in each tensor factor. G leaves R_0 invariant. Denote by R_0^G and R^G the sub-algebras which remain element wise invariant under the action of G. The resulting Jones inclusions $R_0^G \subset R^G$ are consistent with the ADE correspondence.

2. The argument suggests the existence of quantum versions of subgroups of SU(2) for which representations are truncations of those for ordinary subgroups. The results have been generalized to other Lie groups.

3. Also $SL(2, C)$ acts as automorphisms of $M_2(C)$. An interesting question is what happens if one allows G to be any discrete subgroups of SL(2,C). Could this give inclusions with $\mathcal{M} : \mathcal{N} > 4$?. The strong analogy of the spectrum of indices with spectrum of energies with hydrogen atom would encourage this interpretation: the subgroup SL(2,C) not reducing to those of SU(2) would correspond to the possibility for the particle to move with respect to each other with constant velocity.

3.2 The fundamental construction and Temperley-Lieb algebras

It was shown by Jones [41] that for a given Jones inclusion with $\beta = \mathcal{M} : \mathcal{N} < \infty$ there exists a tower of finite II_1 factors $\mathcal{M}_k$ for $k = 0, 1, 2, ...$ such that

1. $\mathcal{M}_0 = \mathcal{N}$, $\mathcal{M}_1 = \mathcal{M}$,
2. $\mathcal{M}_{k+1} = End_{\mathcal{M}_{k-1}}\mathcal{M}_k$ is the von Neumann algebra of operators on $L^2(\mathcal{M}_k)$ generated by $\mathcal{M}_k$ and an orthogonal projection $e_k : L^2(\mathcal{M}_k) \rightarrow L^2(\mathcal{M}_{k-1})$ for $k \geq 1$, where $\mathcal{M}_k$ is regarded as a subalgebra of $\mathcal{M}_{k+1}$ under right multiplication.

It can be shown that $\mathcal{M}_{k+1}$ is a finite factor. The sequence of projections on $\mathcal{M}_\infty = \cup_{k\geq 0}\mathcal{M}_k$ satisfies the relations

$$\begin{array}{ll} e_i^2 = e_i \ , & e_i^= e_i \ , \\ e_i = \beta e_i e_j e_i & \text{for } \ |i-j| = 1 \ , \\ e_i e_j = e_j e_i & \text{for } \ |i-j| \geq 2 \ . \end{array} \tag{4}$$

The construction of hyper-finite II_1 factor using Clifford algebra $C(2)$ represented by 2×2 matrices allows to understand the theorem in $\beta = 4$ case in a straightforward manner. In particular, the second formula involving β follows from the identification of x at $(k-1)^{th}$ level with $(1/\beta)diag(x, x)$ at k^{th} level.

By replacing 2×2 matrices with $\sqrt{\beta} \times \sqrt{\beta}$ matrices one can understand heuristically what is involved in the more general case. $\mathcal{M}_k$ is $\mathcal{M}_{k-1}$ module with dimension $\sqrt{\beta}$ and $\mathcal{M}_{k+1}$ is the space of $\sqrt{\beta} \times \sqrt{\beta}$ matrices $\mathcal{M}_{k-1}$ valued entries acting in $\mathcal{M}_k$. The transition from $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$ linear maps of $\mathcal{M}_k$ happens in the transition to the next level. x at $(k-1)^{th}$ level is identified as $(x/\beta) \times Id_{\sqrt{\beta}\times\sqrt{\beta}}$ at the next level. The projection e_k picks up the projection of the matrix with $\mathcal{M}_{k-1}$ valued entries in the direction of the $Id_{\sqrt{\beta}\times\sqrt{\beta}}$.

The union of algebras $A_{\beta,k}$ generated by $1, e_1, ..., e_k$ defines Temperley-Lieb algebra A_β [43]. This algebra is naturally associated with braids. Addition of one strand to a braid adds one generator to this algebra and the representations of the Temperley Lieb algebra provide link, knot, and 3-manifold invariants [27]. There is also a connection with systems of statistical physics and with Yang-Baxter algebras [44].

A further interesting fact about the inclusion hierarchy is that the elements in $\mathcal{M}_i$ belonging to the commutator $\mathcal{N}'$ of $\mathcal{N}$ form finite-dimensional spaces. Presumably the dimension approaches infinity for $n \rightarrow \infty$.

3.3 Connection with Dynkin diagrams

The possibility to assign Dynkin diagrams ($\beta < 4$) and extended Dynkin diagrams ($\beta = 4$ to Jones inclusions can be understood heuristically by considering a characterization of so called bipartite graphs [42, 39] by the norm of the adjacency matrix of the graph.

Bipartite graphs Γ is a finite, connected graph with multiple edges and black and white vertices such that any edge connects white and black vertex and starts from a white one. Denote by

$w(\Gamma)$ ($b(\Gamma)$) the number of white (black) vertices. Define the adjacency matrix $\Lambda = \Lambda(\Gamma)$ of size $b(\Gamma) \times w(\Gamma)$ by

$$w_{b,w} = \left\{ \begin{array}{ll} m(e) & \mbox{if there exists } e \mbox{ such that } \delta e = b - w \ , \\ 0 & \mbox{otherwise } . \end{array} \right. \quad (5)$$

Here $m(e)$ is the multiplicity of the edge e.

Define norm $||\Gamma||$ as

$$\begin{array}{lll} ||X|| & = & max\{||X||; \ ||x|| \leq 1\} \ , \\ ||\Gamma|| & = & ||\Lambda(\Gamma)|| = \left|\left| \begin{array}{ll} 0 & \Lambda(\Gamma) \\ \Lambda(\Gamma)^t & 0 \end{array} \right|\right| \ . \end{array} \quad (6)$$

Note that the matrix appearing in the formula is $(m+n) \times (m+n)$ symmetric square matrix so that the norm is the eigenvalue with largest absolute value.

Suppose that Γ is a connected finite graph with multiple edges (sequences of edges are regarded as edges). Then

1. If $||\Gamma|| \leq 2$ and if Γ has a multiple edge, $||\Gamma|| = 2$ and $\Gamma = \tilde{A}_1$, the extended Dynkin diagram for $SU(2)$ Kac Moody algebra.

2. $||\Gamma|| < 2$ if and only Γ is one of the Dynkin diagrams of A,D,E. In this case $||\Gamma|| = 2cos(\pi/h)$, where h is the Coxeter number of Γ.

3. $||\Gamma|| = 2$ if and only if Γ is one of the extended Dynkin diagrams $\tilde{A}, \tilde{D}, \tilde{E}$.

This result suggests that one can indeed assign to the Jones inclusions Dynkin diagrams. To really understand how the inclusions can be characterized in terms bipartite diagrams would require a deeper understanding of von Neumann algebras. The following argument only demonstrates that bipartite graphs naturally describe inclusions of algebras.

1. Consider a bipartite graph. Assign to each white vertex linear space $W(w)$ and to each edge of a linear space $W(b,w)$. Assign to a given black vertex the vector space $\oplus_{\delta e = b-w} W(b,w) \otimes W(w)$ where (b,w) corresponds to an edge ending to b.

2. Define $\mathcal{N}$ as the direct sum of algebras $End(W(w))$ associated with white vertices and $\mathcal{M}$ as direct sum of algebras $\oplus_{\delta e = b-w} End(W(b,w)) \otimes End(W(w))$ associated with black vertices.

3. There is homomorphism $N \rightarrow M$ defined by imbedding direct sum of white endomorphisms x to direct sum of tensor products x with the identity endomorphisms associated with the edges starting from x.

It is possible to show that Jones inclusions correspond to the Dynkin diagrams of A_n, D_{2n}, and E_6, E_8 and extended Dynkin diagrams of ADE type. In particular, the dual of the bi-partite graph associated with $\mathcal{M}_{n-1} \subset \mathcal{M}_n$ obtained by exchanging the roles of white and black vertices describes the inclusion $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ so that two subsequent Jones inclusions might define something fundamental (the corresponding space-time dimension is $2 \times log_2(\mathcal{M} : \mathcal{N}) \leq 4$.

3.4 Indices for the inclusions of type III_1 factors

Type III_1 factors appear in relativistic quantum field theory defined in 4-dimensional Minkowski space [35]. An overall summary of basic results discovered in algebraic quantum field theory is described in the lectures of Longo [20]. In this case the inclusions for algebras of observables are induced by the inclusions for bounded regions of M^4 in axiomatic quantum field theory. Tomita's theory of modular Hilbert algebras [33, 36] forms the mathematical corner stone of the theory.

The basic notion is Haag-Kastler net [51] consisting of bounded regions of M^4. Double cone serves as a representative example. The von Neumann algebra $\mathcal{A}(O)$ is generated by observables localized in bounded region O. The net satisfies the conditions implied by local causality:

1. Isotony: $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
2. Locality: $O_1 \subset O_2'$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$ with O' defined as $\{x : \langle x, y\rangle < 0 \ \text{ for all } \ y \in O\}$.
3. Haag duality $\mathcal{A}(O')' = \mathcal{A}(O)$.

   Besides this Poincare covariance, positive energy condition, and the existence of vacuum state is assumed.

DHR (Doplicher-Haag-Roberts) [52] theory allows to deduce the values of Jones index and they are squares of integers in dimensions $D > 2$ so that the situation is rather trivial. The 2-dimensional case is distinguished from higher dimensional situations in that braid group replaces permutation group since the paths representing the flows permuting identical particles can be linked in $X^2 \times T$ and anyonic statistics [46, 47] becomes possible. In the case of 2-D Minkowski space M^2 Jones inclusions with $\mathcal{M} : \mathcal{N} < 4$ plus a set of discrete values of $\mathcal{M} : \mathcal{N}$ in the range $(4, 6)$ are possible. In [20] some values are given ($\mathcal{M} : \mathcal{N} = 5, 5.5049..., 5.236...., 5.828...$).

At least intersections of future and past light cones seem to appear naturally in TGD framework such that the boundaries of future/past directed light cones serve as seats for incoming/outgoing states defined as intersections of space-time surface with these light cones. III_1 sectors cannot thus be excluded as factors in TGD framework. On the other hand, the construction of S-matrix at space-time level is reduced to II_1 case by effective 2-dimensionality.

4 TGD and hyper-finite factors of type II_1: ideas and questions

By effective 2-dimensionality of the construction of quantum states the hyper-finite factors of type II_1 fit naturally to TGD framework. In particular, infinite dimensional spinors define a canonical representations of this kind of factor. In the following the general ideas about factors of type II_1 are discussed and various questions are raised.

4.1 Problems associated with the physical interpretation of III_1 factors

Algebraic quantum field theory approach [34, 35] has led to a considerable understanding of relativistic quantum field theories in terms of hyper-finite III_1 factors. There are however several reasons to suspect that the resulting picture is in conflict with physical intuition. Also the infinities of non-trivial relativistic QFTs suggest that something goes wrong.

4.1.1 Are the infinities of quantum field theories due the wrong type of von Neumann algebra?

The infinities of quantum field theories involve basically infinite traces and it is now known that the algebras of observables for relativistic quantum field theories for bounded regions of Minkowski space correspond to hyper-finite III_1 algebras, for which non-trivial traces are always infinite. This might be the basic cause of the divergence problems of relativistic quantum field theory.

On basis of this observations there is some temptation to think that the finite traces of hyper-finite II_1 algebras might provide a resolution to the problems but not necessarily in QFT context. One can play with the thought that the subtraction of infinities might be actually a process in which III_1 algebra is transformed to II_1 algebra. A more plausible idea suggested by dimensional regularization is that the elimination of infinities actually gives rise to II_1 inclusion at the limit $\mathcal{M} : \mathcal{N} \rightarrow 4$. It is indeed known that the dimensional regularization procedure of quantum field theories can be formulated in terms of bi-algebras assignable to Feynman diagrams and [48, 49] and the emergence of bi-algebras suggests that a connection with II_1 factors and critical role of dimension $D = 4$ might exist.

4.1.2 Continuum of inequivalent representations of commutation relations

There is also a second difficulty related to type III algebras. There is a continuum of inequivalent representations for canonical commutation relations [50]. In thermodynamics this is blessing since temperature parameterizes these representations. In quantum field theory context situation is however different and this problem has been usually put under the rug.

4.1.3 Entanglement and von Neumann algebras

In quantum field theories where 4-D regions of space-time are assigned to observables. In this case hyper-finite type III_1 von Neumann factors appear. Also now inclusions make sense and has been studied: in fact, the parameters characterizing Jones inclusions appear also now and this due to the very general properties of the inclusions.

The algebras of type III_1 have rather counter-intuitive properties from the point of view of entanglement. For instance, product states between systems having space-like separation are not possible at all so that one can speak of intrinsic entanglement [53]. What looks worse is that the decomposition of entangled state to product states is highly non-unique.

Mimicking the steps of von Neumann one could ask what the notion of observables could mean in TGD framework. Effective 2-dimensionality states that quantum states can be constructed using the data given at partonic or stringy 2-surfaces. This data includes also information about normal derivatives so that 3-dimensionality actually lurks in. In any case this would mean that observables are assignable to 2-D surfaces. This would suggest that hyper-finite II_1 factors appear in quantum TGD at least as the contribution of single space-time surface to S-matrix is considered. The contributions for configuration space degrees of freedom meaning functional (not path-) integral over 3-surfaces could of course change the situation.

Also in case of II_1 factors, entanglement shows completely new features which need not however be in conflict with TGD inspired view about entanglement. The eigen values of density matrices are infinitely degenerate and quantum measurement can remove this degeneracy only partially. TGD inspired theory of consciousness has led to the identification of rational (more generally algebraic entanglement) as bound state entanglement stable in state function reduction. When an infinite number of states are entangled, the entanglement would correspond to rational (algebraic number) valued traces for the projections to the eigen states of the density matrix. The canonical transformations of CP_2 are almost $U(1)$ gauge symmetries broken only by classical gravitation. They imply a gigantic spin glass degeneracy which could be behind the infinite degeneracies of eigen states of density matrices in case of II_1 factors.

4.2 Bott periodicity, its generalization, and dimension $D = 8$ as an inherent property of the hyper-finite II_1 factor

Hyper-finite II_1 factor can be constructed as infinite-dimensional tensor power of the Clifford algebra $M_2(C) = C(2)$ in dimension $D = 2$. More precisely, one forms the union of the Clifford algebras $C(2n) = C(2)^{\otimes n}$ of $2n$-dimensional spaces by identifying the element $x \in C(2n)$ as block diagonal elements $diag(x, x)$ of $C(2(n+1))$. The union of these algebras is completed in weak operator topology and can be regarded as a Clifford algebra of real infinite-dimensional separable Hilbert space and thus as sub-algebra of I_∞. Also generalizations obtained by replacing complex numbers by quaternions and octions are possible.

1. The dimension 8 is an inherent property of the hyper-finite II_1 factor since Bott periodicity theorem states $C(n+8) = C_n(16)$. In other words, the Clifford algebra $C(n+8)$ is equivalent with the algebra of 16×16 matrices with entries in $C(n)$. Or articulating it still differently: $C(n+8)$ can be regarded as 16×16 dimensional module with $C(n)$ valued coefficients. Hence the elements in the union defining the canonical representation of hyper-finite II_1 factor are $16^n \times 16^n$ matrices having $C(0)$, $C(2)$, $C(4)$ or $C(6)$ valued valued elements.

2. The idea about a local variant of the infinite-dimensional Clifford algebra defined by power series of space-time coordinate with Taylor coefficients which are Clifford algebra elements fixes the interpretation. The representation as a linear combination of the generators of Clifford algebra of the finite-dimensional space allows quantum generalization only in the case of Minkowski spaces. However, if Clifford algebra generators are representable as gamma *matrices*, the powers of coordinate can be absorbed to the Clifford algebra and the local algebra is lost. Only if the generators are represented as quantum versions of octonions allowing no matrix representation because of their non-associativity, the local algebra makes sense. From this it is easy to deduce both quantum and classical TGD.

4.3 Is a new kind of Feynman diagrammatics needed?

In light of these arguments, one can ask whether the approach based on Feynman diagrams and path integrals forced by quantum field theory characterized by a heroic and futile attempt to find infinity free quantum field theory, and culminating to string models, might represent a wrong turn in the history of physics.

In TGD classical physics is not assigned to a stationary phase approximation but is an exact part of quantum physics. In particular, the poorly defined path integrals over all possible 4-surfaces do not appear at all, being replaced by a functional integral over configuration space of 3-surfaces with exponent of Kähler function defining a vacuum functional which is a non-local functional of 3-surface so that no local divergences result. Gaussian and metric determinants cancel each other by Kähler property. This suggests a new approach based on the generalization of stringy diagrams. Generalized Feynman diagrams are analogous to representations of computations or analytic continuations so that diagrams with loops are equivalent with tree diagrams. This generalizes the notion of duality of old fashioned string models.

The low-dimensional algebras correspond to II_1 algebras and there is a strong temptation to believe that the effective 2-dimensionality of the state construction in TGD framework implies II_1 algebras. The proposed generalized Feynman diagrammatics at space-time level [C7] is not expected to give rise to quantum field theory limit except as an effective theory. Configuration space spinor fields would naturally correspond to a tensor product of bosonic and fermionic II_∞ factors: actually a tensor product of four II_1 factors is involved by quark-lepton degeneracy. It must be however admitted that the appearance of future and past oriented light cones of M^4 in the basic construction of the theory means that also factors of type III_1 might creep in.

4.4 The interpretation of Jones inclusions in TGD framework

By the basic self-referential property of von Neumann algebras one can consider several interpretations of Jones inclusions consistent with sub-system-system relationship, and it is better to start by considering the options that one can imagine.

4.4.1 How Jones inclusions relate to the new view about sub-system?

Jones inclusion characterizes the imbedding of sub-system $\mathcal{N}$ to $\mathcal{M}$and $\mathcal{M}$ as a finite-dimensional $\mathcal{N}$-module is the counterpart for the tensor product in finite-dimensional context. The possibility to express $\mathcal{M}$ as $\mathcal{N}$ module $\mathcal{M}/\mathcal{N}$ states fractality and can be regarded as a kind of self-referential "Brahman=Atman identity" at the level of infinite-dimensional systems.

Also the mysterious looking almost identity $CH^2 = CH$ for the configuration space of 3-surfaces would fit nicely with the identity $M \oplus M = M$. $M \otimes M \subset M$ in configuration space Clifford algebra degrees of freedom is also implied and the construction of $\mathcal{M}$ as a union of tensor powers of $C(2)$ suggests that $M \otimes M$ allows $\mathcal{M} : \mathcal{N} = 4$ inclusion to $\mathcal{M}$. This paradoxical result conforms with the strange self-referential property of factors of II_1.

The notion of many-sheeted space-time forces a considerable generalization of the notion of sub-system and simple tensor product description is not enough. Topological picture based on the length scale resolution suggests even the possibility of entanglement between sub-systems of un-entangled sub-systems. The possibility that hyper-finite II_1-factors describe the physics of TGD also in bosonic degrees of freedom is suggested by configuration space super-symmetry. On the other hand, bosonic degrees could naturally correspond to I_∞ factor so that hyper-finite II_∞ would be the net result.

The most general view is that Jones inclusion describes all kinds of sub-system-system inclusions. The possibility to assign conformal field theory to the inclusion gives hopes of rather detailed view about dynamics of inclusion.

1. The topological condensation of space-time sheet to a larger space-time sheet mediated by wormhole contacts could be regarded as Jones inclusion. $\mathcal{N}$ would correspond to the condensing space-time sheet, $\mathcal{M}$ to the system consisting of both space-time sheets, and $\sqrt{\mathcal{M} : \mathcal{N}}$ would characterize the number of quantum spinorial degrees of freedom associated with the interaction between space-time sheets. Note that by general results $\mathcal{M} : \mathcal{N}$ characterizes the fractal dimension of quantum group ($\mathcal{M} : \mathcal{N} < 4$) or Kac-Moody algebra ($\mathcal{M} : \mathcal{N} = 4$) [39].

2. The branchings of space-time sheets (space-time surface is thus homologically like branching like of Feynman diagram) correspond naturally to n-particle vertices in TGD framework. What is nice is that vertices are nice 2-dimensional surfaces rather than surfaces having typically pinch singularities. Jones inclusion would naturally appear as inclusion of operator spaces $\mathcal{N}_i$ (essentially Fock spaces for fermionic oscillator operators) creating states at various lines as sub-spaces $N_i \subset M$ of operators creating states in common von Neumann factor $\mathcal{M}$. This would allow to construct vertices and vertices in natural manner using quantum groups or Kac-Moody algebras.

   The fundamental $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \otimes_N \mathcal{M}$ inclusion suggests a concrete representation based on the identification $N_i = M$, where M is the universal Clifford algebra associated with incoming line and $\mathcal{N}$ is defined by the condition that $\mathcal{M}/\mathcal{N}$ is the quantum variant of Clifford algebra of H. N-particle vertices could be defined as traces of Connes products of the operators creating incoming and outgoing states. It will be found that this leads to a master formula for S-matrix if the generalization of the old-fashioned string model duality implying that all generalized Feynman diagrams reduce to diagrams involving only single vertex is accepted.

3. If 4-surfaces can branch as the construction of vertices requires, it is difficult to argue that 3-surfaces and partonic/stringy 2-surfaces could not do the same. As a matter fact, the master formula for S-matrix to be discussed later explains the branching of 4-surfaces as an apparent affect. Despite this one can consider the possibility that this kind of joins are possible so that a new kind of mechanism of topological condensation would become possible. 3-space-sheets and partonic 2-surfaces whose p-adic fractality is characterized by different p-adic primes could be connected by "joins" representing branchings of 2-surfaces. The structures formed by soap film foam provide a very concrete illustration about what would happen. In the TGD based model of hadrons [F4] it has been assumed that join along boundaries bonds (JABs) connect quark space-time space-time sheets to the hadronic space-time sheet. The problem is that, at least for identical primes, the formation of join along boundaries bond fuses two systems to single bound state. If JABs are replaced joins, this objection is circumvented.

4. The space-time correlate for the formation of bound states is the formation of JABs. Standard intuition tells that the number of degrees of freedom associated with the bound state is smaller than the number of degrees of freedom associated with the pair of free systems. Hence the inclusion of the bound state to the tensor product could be regarded as Jones inclusion. On the other hand, one could argue that the JABs carry additional vibrational degrees of freedom so that the idea about reduction of degrees of freedom might be wrong: free system could be regarded as sub-system of bound state by Jones inclusion. The self-referential holographic properties of von Neumann algebras allow both interpretations: any system can be regarded as sub-system of any system in accordance with the bootstrap idea.

5. Maximal deterministic regions inside given space-time sheet bounded by light-like causal determinants define also sub-systems in a natural manner and also their inclusions would naturally correspond to Jones inclusions.

6. The TGD inspired model for topological quantum computation involves the magnetic flux tubes defined by join along boundaries bonds connecting space-time sheets having light-like boundaries. These tubes condensed to background 3-space can become linked and knotted and code for quantum computations in this manner. In this case the addition of new strand to the system corresponds to Jones inclusion in the hierarchy associated with inclusion $\mathcal{N} \subset \mathcal{M}$. The anyon states associated with strands would be represented by a finite tensor product of quantum spinors assignable to $\mathcal{M}/\mathcal{N}$ and representing quantum counterpart of H-spinors.

One can regard $\mathcal{M} : \mathcal{N}$ degrees of freedom correspond to quantum group or Kac-Moody degrees of freedom. Quantum group degrees of freedom relate closely to the conformal and topological degrees of freedom as the connection of II_1 factors with topological quantum field theories and braid matrices suggests itself. For the canonical inclusion this factorization would correspond to factorization of quantum H-spinor from configuration space spinor.

A more detailed study of canonical inclusions to be carried out later demonstrates what this factorization corresponds at the space-time level to a formation of space-time sheets which can be regarded as multiple coverings of M^4 and CP_2 with invariance group $G = G_a \times G_b \subset SL(2,C) \times SU(2)$, $SU(2) \subset SU(3)$. The unexpected outcome is that Planck constants assignable to M^4 and CP_2 degrees of freedom depend on the canonical inclusions. The existence of macroscopic quantum phases with arbitrarily large Planck constants is predicted.

It would seem possible to assign the $\mathcal{M} : \mathcal{N}$ degrees quantum spinorial degrees of freedom to the interface between subsystems represented by $\mathcal{N}$ and $\mathcal{M}$. The interface could correspond to the wormhole contacts, joins, JABs, or light-like causal determinants serving as boundary between maximal deterministic regions, etc... In terms of the bipartite diagrams representing the inclusions, joins (say) would correspond to the edges connecting white vertices representing sub-system (the entire system without the joins) to black vertices (entire system).

4.4.2 About the interpretation of $\mathcal{M}:\mathcal{N}$ degrees of freedom

The Clifford algebra $\mathcal{N}$ associated with a system formed by two space-time sheet can be regarded as $1 \leq \mathcal{M}:\mathcal{N} \leq 4$-dimensional module having $\mathcal{N}$ as its coefficients. It is possible to imagine several interpretations the degrees of freedom labelled by β.

1. The $\beta = \mathcal{M}:\mathcal{N}$ degrees of freedom could relate to the interaction of the space-time sheets. Beraha numbers appear in the construction of S-matrices of topological quantum field theories and an interpretation in terms of braids is possible. This would suggest that the interaction between space-time sheets can be described in terms of conformal quantum field theory and the S-matrices associated with braids describe this interaction. Jones inclusions would characterize the effective number of active conformal degrees of freedom. At $n = 3$ limit these degrees of freedom disappear completely since the conformal field theory defined by the Chern-Simons action describing this interaction would become trivial ($c = 0$ as will be found).

2. The interpretation in terms of imbedding space Clifford algebra would suggest that β-dimensional Clifford algebra of $\sqrt{\beta}$-dimensional spinor space is in question. For $\beta = 4$ the algebra would be the Clifford algebra of 2-dimensional space. $\mathcal{M}/\mathcal{N}$ would have interpretation as complex quantum spinors with components satisfying $z_1z_2 = qz_2z_1$ and its conjugate and having fractal complex dimension $\sqrt{\beta}$. This would conform with the effective 2-dimensionality of TGD. For $\beta < 4$ the fractal dimension of partonic quantum spinors defining the basic conformal fields would be reduced and become $d = 1$ for $n = 3$: the interpretation is in terms of strong correlations caused by the non-commutativity of the components of quantum spinor. For number theoretical generalizations of infinite-dimensional Clifford algebras $Cl(C)$ obtained by replacing C with Abelian complexification of quaternions or octonions one would obtain higher-dimensional spinors.

4.5 Configuration space, space-time, and imbedding space and hyper-finite type II_1 factors

The preceding considerations have by-passed the question about the relationship of the configuration space tangent space to its Clifford algebra. Also the relationship between space-time and imbedding space and their quantum variants could be better. In particular, one should understand how effective 2-dimensionality can be consistent with the 4-dimensionality of space-time.

4.5.1 Super-conformal symmetry and configuration space Poisson algebra as hyper-finite type II_1 factor

It would be highly desirable to achieve also a description of the configuration space degrees of freedom using von Neumann algebras. Super-conformal symmetry relating fermionic degrees of freedom and configuration space degrees of freedom suggests that this might be the case. Super-canonical algebra has as its generators configuration space Hamiltonians and their super-counterparts identifiable as CH gamma matrices. Super-symmetry requires that the Clifford algebra of CH and the Hamiltonian vector fields of CH with symplectic central extension both define hyper-finite II_1 factors. By super-symmetry Poisson bracket corresponds to an anti-commutator for gamma matrices. The ordinary quantized version of Poisson bracket is obtained as $\{P_i, Q_j\} \rightarrow [P_i, Q_j] = J_{ij}Id$. Finite trace version results by assuming that Id corresponds to the projector CH Clifford algebra having unit norm. The presence of zero modes means direct integral over these factors.

Configuration space gamma matrices anti-commuting to identity operator with unit norm corresponds to the tangent space $T(CH)$ of CH. Thus it would be not be surprising if $T(CH)$ could be imbedded in the sigma matrix algebra as a sub-space of operators defined by the gamma matrices generating this algebra. At least for $\beta = 4$ construction of hyper-finite II_1 factor this definitely makes sense.

The dimension of the configuration space defined as the trace of the projection operator to the sub-space spanned by gamma matrices is obviously zero. Thus configuration space has in this sense the dimensionality of single space-time point. This sounds perhaps absurd but the generalization of the number concept implied by infinite primes indeed leads to the view that single space-time point is infinitely structured in the number theoretical sense although in the real sense all states of the point are equivalent [E10]. The reason is that there is infinitely many numbers expressible as ratios of infinite integers having unit real norm in the real sense but having different p-adic norms.

4.5.2 How to understand the dimensions of space-time and imbedding space?

One should be able to understand the dimensions of 3-space, space-time and imbedding space in a convincing matter in the proposed framework. There is also the question whether space-time and imbedding space emerge uniquely from the mathematics of von Neumann algebras alone.

1. The dimensions of space-time and imbedding space

Two sub-sequent inclusions dual to each other define a special kind of inclusion giving rise to a quantum counterpart of $D = 4$ naturally. This would mean that space-time is something which emerges at the level of cognitive states.

The special role of classical division algebras in the construction of quantum TGD [E2], $D = 8$ Bott periodicity generalized to quantum context, plus self-referential property of type II_1 factors might explain why 8-dimensional imbedding space is the only possibility.

State space has naturally quantum dimension $D \leq 8$ as the following simple argument shows. The space of quantum states has quark and lepton sectors which both are super-symmetric implying $D \leq 4$ for each. Since these sectors correspond to different Hamiltonian algebras (triality one for quarks and triality zero for leptonic sector), the state space has quantum dimension $D \leq 8$.

2. How the lacking two space-time dimensions emerge?

3-surface is the basic dynamical unit in TGD framework. This seems to be in conflict with the effective 2-dimensionality [E2] meaning that partonic 2-surface code for quantum states, and with the fact that hyper-finite II_1 factors have intrinsic quantum dimension 2.

A possible resolution of the problem is that the foliation of 3-surface by partonic two-surfaces defines a one-dimensional direct integral of isomorphic hyper-finite type II_1 factors, and the zero mode labelling the 2-surfaces in the foliation serves as the third spatial coordinate. For a given 3-surface the contribution to the configuration space metric can come only from 2-D partonic surfaces defined as intersections of 3-D light-like CDs with $X^7_\pm$ [B2, B3]. Hence the direct integral should somehow relate to the classical non-determinism of Kähler action.

1. The one-parameter family of intersections of light-like CD with $X^7_\pm$ inside $X^4 \cap X^7_\pm$ could indeed be basically due to the classical non-determinism of Kähler action. The contribution to the metric from the normal light-like direction to $X^3 = X^4 \cap X^7_\pm$ can cause the vanishing of the metric determinant $\sqrt{g_4}$ of the space-time metric at $X^2 \subset X^3$ under some conditions on X^2. This would mean that the space-time surface $X^4(X^3)$ is not uniquely determined by the minimization principle defining the value of the Kähler action, and the complete dynamical specification of X^3 requires the specification of partonic 2-surfaces X^2_i with $\sqrt{g_4} = 0$.
2. The known solutions of field equations [D1] define a double foliation of the space-time surface defined by Hamilton-Jacobi coordinates consisting of complex transversal coordinate and two light-like coordinates for M^4 (rather than space-time surface). Number theoretical considerations inspire the hypothesis that this foliation exists always [E2]. Hence a natural hypothesis

is that the allowed partonic 2-surfaces correspond to the 2-surfaces in the restriction of the double foliation of the space-time surface by partonic 2-surfaces to X^3, and are thus locally parameterized by single parameter defining the third spatial coordinate.

3. There is however also a second light-like coordinate involved and one might ask whether both light-like coordinates appear in the direct sum decomposition of II_1 factors defining $T(CH)$. The presence of two kinds of light-like CDs would provide the lacking two space-time coordinates and quantum dimension $D = 4$ would emerge at the limit of full non-determinism. Note that the duality of space-like partonic and light-like stringy 2-surfaces conforms with this interpretation since it corresponds to a selection of partonic/stringy 2-surface inside given 3-D CD whereas the dual pairs correspond to different CDs.

4. That the quantum dimension would be $2D_q = \beta < 4$ above CP_2 length scale conforms with the fact that non-determinism is only partial and time direction is dynamically frozen to a high degree. For vacuum extremals there is strong non-determinism but in this case there is no real dynamics. For CP_2 type extremals, which are not vacuum extremals as far action and small perturbations are considered, and which correspond to $\beta = 4$ there is a complete non-determinism in time direction since the M^4 projection of the extremal is a light-like random curve and there is full 4-D dynamics. Light-likeness gives rise to conformal symmetry consistent with the emergence of Kac Moody algebra [D1].

3. Time and cognition

In a completely deterministic physics time dimension is strictly speaking redundant since the information about physical states is coded by the initial values at 3-dimensional slice of space-time. Hence the notion of time should emerge at the level of cognitive representations possible by to the non-determinism of the classical dynamics of TGD.

Since Jones inclusion means the emergence of cognitive representation, the space-time view about physics should correspond to cognitive representations provided by Feynman diagram states with zero energy with entanglement defined by a two-sided projection of the lowest level S-matrix. These states would represent the "laws of quantum physics" cognitively. Also space-time surface serves as a classical correlate for the evolution by quantum jumps with maximal deterministic regions serving as correlates of quantum states. Thus the classical non-determinism making possible cognitive representations would bring in time. The fact that quantum dimension of space-time is smaller than $D = 4$ would reflect the fact that the loss of determinism is not complete.

4. Do space-time and imbedding space emerge from the theory of von Neumann algebras and number theory?

The considerations above force to ask whether the notions of space-time and imbedding space emerge from von Neumann algebras as predictions rather than input. The fact that it seems possible to formulate the S-matrix and its generalization in terms of inherent properties of infinite-dimensional Clifford algebras suggest that this might be the case.

4.6 Quaternions, octonions, and hyper-finite type II_1 factors

Quaternions and octonions as well as their hyper counterparts obtained by multiplying imaginary units by commuting $\sqrt{-1}$ and forming a sub-space of complexified division algebra, are in in a central role in the number theoretical vision about quantum TGD [E2]. Therefore the question arises whether complexified quaternions and perhaps even octonions could be somehow inherent properties of von Neumann algebras. One can also wonder whether the quantum counterparts of quaternions and octonions could emerge naturally from von Neumann algebras. The following considerations allow to get grasp of the problem.

4.6.1 Quantum quaternions and quantum octonions

Quantum quaternions have been constructed as deformation of quaternions [57]. The key observation that the Glebsch Gordan coefficients for the tensor product $3 \otimes 3 = 5 \oplus \oplus 3 \oplus 1$ of spin 1 representation of $SU(2)$ with itself gives the anti-commutative part of quaternionic product as spin 1 part in the decomposition whereas the commutative part giving spin 0 representation is identifiable as the scalar product of the imaginary parts. By combining spin 0 and spin 1 representations, quaternionic product can be expressed in terms of Glebsh-Gordan coefficients. By replacing GGC:s by their quantum group versions for group $sl(2)_q$, one obtains quantum quaternions.

There are two different proposals for the construction of quantum octonions [58, 59]. Also now the idea is to express quaternionic and octonionic multiplication in terms of Glebsch-Gordan coefficients and replace them with their quantum versions.

1. The first proposal [58] relies on the observation that for the tensor product of $j = 3$ representations of $SU(2)$ the Glebsch-Gordan coefficients for $7 \otimes 7 \rightarrow 7$ in $7 \otimes 7 = 9 \oplus 7 \oplus 5 \oplus 3 \oplus 1$ defines a product, which is equivalent with the antisymmetric part of the product of octonionic imaginary units. As a matter fact, the antisymmetry defines 7-dimensional Malcev algebra defined by the anticommutator of octonion units and satisfying b definition the identity

$$[[x, y, z], x] = [x, y, [x, z]] \quad , \quad [x, y, z] \equiv [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \quad . \tag{7}$$

   7-element Malcev algebra defining derivations of octonionic algebra is the only complex Malcev algebra not reducing to a Lie algebra. The $j = 0$ part of the product corresponds also now to scalar product for imaginary units. Octonions are constructed as sums of $j = 0$ and $j = 3$ parts and quantum Glebsch-Gordan coefficients define the octonionic product.

2. In the second proposal [59] the quantum group associated with $SO(8)$ is used. This representation does not allow unit but produces a quantum version of octonionic triality assigning to three octonions a real number.

4.6.2 Quaternionic or octonionic quantum mechanics?

There have been numerous attempts to introduce quaternions and octonions to quantum theory. Quaternionic or octonionic quantum mechanics, which means the replacement of the complex numbers as coefficient field of Hilbert space with quaternions or octonions, is the most obvious approach (for example and references to the literature see for instance [56].

In both cases non-commutativity poses serious interpretational problems. In the octonionic case the non-associativity causes even more serious obstacles [55, 56].

1. Assuming that an orthonormalized state basis with respect to an octonion valued inner product has been found, the multiplication of any basis with octonion spoils the orthonormality. The proposal to circumvent this difficulty discussed in [55] eliminates non-associativity by assuming that octonions multiply states one by one (rather than multiplying each other before multiplying the state). Effectively this means that octonions are replaced with 8×8-matrices.

2. The definition of the tensor product leads also to difficulties since associativity is lost (recall that Yang-Baxter equation codes for associativity in case of braid statistics [28, 29]).

3. The notion of hermitian conjugation is problematic and forces a selection of a preferred imaginary unit, which does not look nice. Note however that the local selection of a preferred

imaginary unit is in a key role in the proposed construction of space-time surfaces as hyper-quaternionic or co-hyper-quaternionic surfaces and allows to interpret space-time surfaces either as surfaces in 8-D Minkowski space M^8 of hyper-octonions or in $M^4 \times CP_2$. This selection turns out to have quite different interpretation in the proposed framework.

4.6.3 Hyper-finite factor II_1 has a natural Hyper-Kähler structure

In the case of hyper-finite factors of type II_1 quaternions a more natural approach is based on the generalization of the Hyper-Kähler structure rather than quaternionic quantum mechanics. The reason is that also configuration space tangent space should and is expected to have this structure [B3]. The Hilbert space remains a complex Hilbert space but the quaternionic units are represented as operators in Hilbert space. The selection of the preferred unit is necessary and natural. The identity operator representing quaternionic real unit has trace equal to one, is expected to give rise to the series of quantum quaternion algebras in terms of inclusions $\mathcal{N} \subset \mathcal{M}$ having interpretation as N-modules.

The representation of the quaternion units is rather explicit in the structure of hyper-finite II_1 factor. The $\mathcal{M} : \mathcal{N} \equiv \beta = 4$ hierarchical construction can be regarded as Connes tensor product of infinite number of 4-D Clifford algebras of Euclidian plane with Euclidian signature of metric ($diag(-1,-1)$). This algebra is nothing but the quaternionic algebra in the representation of quaternionic imaginary units by Pauli spin matrices multiplied by i.

The imaginary unit of the underlying complex Hilbert space must be chosen and there is whole sphere S^2 of choices and in every point of configuration space the choice can be made differently. The space-time correlate for this local choice of preferred hyper-octonionic unit [E2]. At the level of configuration space geometry the quaternion structure of the tangent space means the existence of Hyper-Kähler structure guaranteing that configuration space has a vanishing Einstein tensor. It it would not vanish, curvature scalar would be infinite by symmetric space property (as in case of loop spaces) and induce a divergence in the functional integral over 3-surfaces from the expansion of $\sqrt{g}$ [B3].

The quaternionic units for the II_1 factor, are simply limiting case for the direct sums of 2×2 units normalized to one. Generalizing from $\beta = 4$ to $\beta < 4$, the natural expectation is that the representation of the algebra as $\beta = \mathcal{M} : \mathcal{N}$-dimensional $\mathcal{N}$-module gives rise to quantum quaternions with quaternion units defined as infinite sums of $\sqrt{\beta} \times \sqrt{\beta}$ matrices.

At Hilbert space level one has an infinite Connes tensor product of 2-component spinor spaces on which quaternionic matrices have a natural action. The tensor product of Clifford algebras gives the algebra of 2×2 quaternionic matrices acting on 2-component quaternionic spinors (complex 4-component spinors). Thus double inclusion could correspond to (hyper-)quaternionic structure at space-time level. Note however that the correspondence is not complete since hyper-quaternions appear at space-time level and quaternions at Hilbert space level.

4.6.4 Von Neumann algebras and octonions

The octonionic generalization of the Hyper-Kähler manifold does not make sense as such since octonionic units are not representable as linear operators. The allowance of anti-linear operators inherently present in von Neumann algebras could however save the situation. Indeed, the Cayley-Dickson construction for the division algebras (for a nice explanation see [60]), which allows to extend any * algebra, and thus also any von Neumann algebra, by adding an imaginary unit it and identified as *, comes in rescue.

The basic idea of the Cayley-Dickson construction is following. The * operator, call it J, representing a conjugation defines an *anti-linear* operator in the original algebra A. One can extend A by adding this operator as a new element to the algebra. The conditions satisfied by J are

$$a(Jb) = J(a^*b) \ , \quad (aJ)b = (ab^*)J \ , \quad (Ja)(bJ^{-1}) = (ab)^* \ . \tag{8}$$

In the associative case the conditions are equivalent to the first condition.

It is intuitively clear that this addition extends the hyper-Kähler structure to an octonionic structure at the level of the operator algebra. The quantum version of the octonionic algebra is fixed by the quantum quaternion algebra uniquely and is consistent with the Cayley-Dickson construction. It is not clear whether the construction is equivalent with either of the earlier proposals [58, 59]. It would however seem that the proposal is simpler.

4.6.5 Physical interpretation of quantum octonion structure

Without further restrictions the extension by J would mean that vertices contain operators, which are superpositions of linear and anti-linear operators. This would give superpositions of states and their time-reversals and mean that state could be a superposition of states with opposite values of say fermion numbers. The problem disappears if either the linear operators A or anti-linear operators JA can be used to construct physical states from vacuum. The fact, that space-time surfaces are either hyper-quaternionic or co-hyper-quaternionic, is a space-time correlate for this restriction.

The $HQ-coHQ$ duality discussed in [E2] states that the descriptions based on hyper-quaternionic and co-hyper-quaternionic surfaces are dual to each other. The duality can have two meanings.

1. The vacuum is invariant under J so that one can use either complexified quaternionic operators A or their co-counterparts of form JA to create physical states from vacuum.

2. The vacuum is not invariant under J. This could relate to the breaking of CP and T invariance known to occur in meson-antimeson systems. In TGD framework two kinds of vacua are predicted corresponding intuitively to vacua in which either the product of all positive or negative energy fermionic oscillator operators defines the vacuum state, and these two vacua could correspond to a vacuum and its J conjugate, and thus to positive and negative energy states. In this case the two state spaces would not be equivalent although the physics associated with them would be equivalent.

The considerations of [E2] related to the detailed dynamics of $HQ-coHQ$ duality demonstrate that the variational principles defining the dynamics of hyper-quaternionic and co-hyper-quaternionic space-time surfaces are antagonistic and correspond to world as seen by a conscientous book-keeper on one hand and an imaginative artist on the other hand. HQ case is conservative: differences measured by the magnitude of Kähler action tend to be minimized, the dynamics is highly predictive, and minimizes the classical energy of the initial state. $coHQ$ case is radical: differences are maximized (this is what the construction of sensory representations would require). The interpretation proposed in [E2] was that the two space-time dynamics are just different predictions for what would happen (has happened) if no quantum jumps would occur (had occurred). A stronger assumption is that these two views are associated with systems related by time reversal symmetry.

What comes in mind first is that this antagonism follows from the assumption that these dynamics are actually time-reversals of each other with respect to M^4 time (the rapid elimination of differences in the first dynamics would correspond to their rapid enhancement in the second dynamics). This is not the case so that T and CP symmetries are predicted to be broken in accordance with the CP breaking in meson-antimeson systems [F5] and cosmological matter-antimatter asymmetry [D5].

4.7 How does the hierarchy of infinite primes relate to the hierarchy of II_1 factors?

The hierarchy of Feynman diagrams accompanying the hierarchy defined by Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset ...$ gives a concrete representation for the hierarchy of cognitive dynamics providing a representation for the material world at the lowest level of the hierarchy. This hierarchy seems to relate directly to the hierarchy of space-time sheets.

Also the construction of infinite primes [E3] leads to an infinite hierarchy. Infinite primes at the lowest level correspond to polynomials of single variable x_1 with rational coefficients, next level to polynomials x_1 for which coefficients are rational functions of variable x_2, etc... so that a natural ordering of the variables is involved.

If the variables x_i are hyper-octonions (subs-space of complexified octonions for which elements are of form $x + \sqrt{-1}y$, where x is real number and y imaginary octonion and $\sqrt{-1}$ is commuting imaginary unit, this hierarchy of states could provide a realistic representation of physical states as far as quantum numbers related to imbedding space degrees of freedom are considered in M^8 picture dual to $M^4 \times CP_2$ picture [E2]. Infinite primes are mapped to space-time surfaces in a manner analogous to the mapping of polynomials to the loci of their zeros so that infinite primes, integers, and rationals become concrete geometrical objects.

Infinite primes are also obtained by a repeated second quantization of a super-symmetric arithmetic quantum field theory. Infinite rational numbers correspond in this description to pairs of positive energy and negative energy states of opposite energies having interpretation as pairs of initial and final states so that higher level states indeed represent transitions between the states. For these reasons this hierarchy has been interpreted as a correlate for a cognitive hierarchy coding information about quantum dynamics at lower levels. This hierarchy has also been assigned with the hierarchy of space-time sheets. Just as the hierarchy of generalized Feynman diagrams provides self representations of the lowest matter level and is coded by it, finite primes code the hierarchy of infinite primes.

Infinite primes, integers, and rationals have finite p-adic norms equal to 1, and one can wonder whether a Hilbert space like structure with dimension given by an infinite prime or integer makes sense, and whether it has anything to do with the Hilbert space for which dimension is infinite in the sense of the limiting value for a dimension of sub-space. The Hilbert spaces with dimension equal to infinite prime would define primes for the tensor product of these spaces. The dimension of this kind of space defined as any p-adic norm would be equal to one.

One cannot exclude the possibility that infinite primes could express the infinite dimensions of hyper-finite III_1 factors, which cannot be excluded and correspond to that part of quantum TGD which relates to the imbedding space rather than space-time surface. Indeed, infinite primes code naturally for the quantum numbers associated with the imbedding space. Secondly, the appearance of 7-D light-like causal determinants $X^7_\pm = M^4_\pm \times CP_2$ forming nested structures in the construction of S-matrix brings in mind similar nested structures of algebraic quantum field theory [35]. If this is were the case, the hierarchy of Beraha numbers possibly associated with the phase resolution could correspond to hyper-finite factors of type II_1, and the decomposition of space-time surface to regions labelled by p-adic primes and characterized by infinite primes could correspond to hyper-finite factors of type III_1 and represent imbedding space degrees of freedom.

The state space would in this picture correspond to the tensor products of hyper-finite factors of type II_1 and III_1 (of course, also factors I_n and I_∞ are also possible). III_1 factors could be assigned to the sub-configuration spaces defined by 3-surfaces in regions of M^4 expressible in terms of unions and intersections of $X^7_\pm = M^4_\pm \times CP_2$. By conservation of four-momentum, bounded regions of this kind are possible only for the states of zero net energy appearing at the higher levels of hierarchy. These sub-configuration spaces would be characterized by the positions of the tips of light cones $M^4_\pm \subset M^4$ involved. This indeed brings in continuous spectrum of four-momenta forcing to introduce non-separable Hilbert spaces for momentum eigen states and necessitating

III_1 factors. Infinities would be avoided since the dynamics proper would occur at the level of space-time surfaces and involve only II_1 factors.

5 Space-time as surface of $M^4 \times CP_2$ and inclusions of hyper-finite type II_1 factors

Double Jones inclusion plays a pivotal role in the theory of von Neumann algebras. Double inclusion $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$, where $\mathcal{M}_1 = \mathcal{M} \otimes_N \mathcal{M}$ is Connes tensor product obtained by identifying the elements $nm_1 \otimes m_2$ and $m_1 \otimes m_2 n$ so that multiplication by $\mathcal{N}$ from left is equivalent to that from right. This double inclusion extends to an infinite hierarchy of inclusions.

These two inclusions are dual and the values of $\mathcal{M} : \mathcal{N}$ are same. One can write M_1 as N module of $\mathcal{M}/\mathcal{N} \otimes \mathcal{M}/\mathcal{N}$ so that one has quantal counterpart of $2d$-spinors associated with 4-D manifolds. This suggests an interpretation of the Jones inclusion $\mathcal{M} \subset \mathcal{M}_1$ as a quantal representation for an imbedding of 2-D manifold into 4-D manifold at the level of spinors regarded having $\mathcal{N}$-valued components.

This is however not quite enough: in TGD one has imbeddings of 4-D space-times to 8-D imbedding space, and one can ask whether also these imbeddings could be represented using Jones inclusions. Quaternionic and octonionic matrix algebras might provide as solution to the problem.

This encourages to ask whether fundamental physics could somehow reduce to Jones inclusions of infinite-dimensional Clifford algebras induced by the inclusion sequence $C \subset H \subset O$ for the classical number fields and by its hyper-counterpart as suggested by number theoretic vision [E2]. In other words, does the notion of space-time as a surface of $M^4 \times CP_2$ emerge automatically from number theoretical infinite-dimensional Clifford algebras? The following arguments try to demonstrate that the generalization of the number theoretical infinite-dimensional Clifford algebras by making them local algebras with respect to hyper-F ($F = C, H, O$) could realize the dream.

5.1 Jones inclusion as a representation for the imbedding $X^4 \subset M^4 \times CP_2$?

The obvious guess is that the complex matrix algebra $M_2(C)$ as a building block of an infinite-dimensional Clifford algebra must be replaced with the quaternionic matrix algebra $M_2(H)$. This is possible since $M_2(H) \otimes_C M_2(H)$ can be defined using Connes tensor product to guarantee that the multiplication by complex numbers from left in the tensor product is equivalent with the multiplication from right. For octonions this construction fails unless one allows non-associativity which is not a problem mathematically but poses interpretational problems.

In this case the quantum spinors associated with $\mathcal{M}/\mathcal{N}$ would have 4 complex components and correspond to 4-D space-time. The quantum spinors associated with $\mathcal{M}_1/\mathcal{N}$ would have 16 complex components and correspond to 8-D space-time. This suggests that $\mathcal{M} \subset \mathcal{M}_1$ represents the imbedding of the space-time surface to the imbedding space: the problems with signature are avoided since spinors are used. $\mathcal{N} \subset \mathcal{M}$ could in turn be interpreted as a representation of the imbedding of 2-dimensional partonic surfaces obtained as a cross section of light-like causal determined into X^4. $\mathcal{M}$ ($\mathcal{M}_1$) could in turn be interpreted in terms of $\mathcal{N}$-valued quantization of space-time (imbedding space) spinors. Any quaternionic double imbedding with $\mathcal{N} : \mathcal{M} \leq 4$ would have the same interpretation so that the space-time and imbedding space dimension would be universal.

This picture is however not yet quite satisfactory. Number theoretic vision suggests that also octonions are important. This suggests that the double inclusion $N \subset M \subset M_1$ should be replaced with a purely number theoretic Jones inclusions induced by the inclusion $C \subset H \subset O$.

5.2 Why $X^4 \subset M^4 \times CP_2$?

The number theoretic vision [E2] interprets space-time surfaces as hyper-quaternionic sub-manifolds of hyper-octonionic imbedding space. The previous considerations would in turn suggests that physical Clifford algebra could be seen as a quaternionic sub-algebra of octonionic algebra by restricting the octonionic coefficients of the Clifford basis to be quaternionic. Thus it would seem that complexified quaternions and octonions are needed and that Clifford algebra degrees should correspond to quaternions-/octonions and bosonic center of mass degrees of freedom to hyper-quaternions/-octonions. In the following an attempt to complete these observations to a coherent picture is made.

5.2.1 CP_2 parameterizes quaternionic sub-factors

Formally the notion of Connes tensor product generalizes to the octonionic context. Strictly speaking, the non-associativity of the matrix multiplication means leaving the framework of von Neumann algebras and could lead to interpretational difficulties although the products as such are completely well defined.

In accordance with the number theoretical vision one might hope that the basic laws of physics laws would result as a resolution of these interpretational difficulties. Quaternionic Clifford algebra indeed emerges naturally as a subalgebra obtained by restricting the octonionic matrices to have quaternion-valued elements. Quaternion-octonion inclusion could thus define the fundamental Jones inclusion at the level of configuration space Clifford algebra. This inclusion could be completed to a double inclusion corresponding to $C \subset H \subset O$.

Remarkably, the set of these subalgebras is parameterized by $CP_2 = SU(3)/U(2)$ since $SU(3)$ is the group of (hyper)-octonionic automorphisms and $U(2)$ leaves a given quaternionic plane invariant. The necessity to select of this sub-algebra would follow from the associativity requirement [E2, E3]. This also conforms with the fact that complex CP_2 coordinates behave locally like $U(2)$ spinor.

5.2.2 M^4 parameterizes the tips of hyper-quaternionic light cones

The next question is how to obtain M^4 factor and how the identification space-time surfaces as hyper-quaternionic sub-manifolds of hyper-octonionic imbedding space [E2] could emerge from this picture.

Consider first what can be regarded as understood.

1. Future and past light-cones appear naturally in the construction of the geometry of the world of the classical worlds (configuration space). Configuration space can be regarded as a union of configuration spaces associated with the future and past light-cones. Therefore the points m of M^4 would appear as moduli labelling the sub-configuration spaces $CH_{+-}(m)$ consisting of surfaces in the union $M^4_{+-} = M^4_+ \cup M^4_-$ in the union $\cup_m CH_{+-}(m)$ defining the configuration space.

2. There is an analogy with conformal field theories. Configuration space spinor fields depend on configuration space coordinates, in particular moduli characterizing the position of light cone cosmology CH_{+-} in decomposition $\cup_m CH_{+-}(m)$. The dependence on m is completely analogous to the dependence of conformal fields on complex coordinate z in conformal fields and induces corresponding dependence on quantum states created by the CH spinor fields. Once this dependence is known, it is possible to calculate products of configuration space spinors fields associated with different light cones M^4_{+-} using operator product expansions in complete analogy with conformal field theories. The M^4 dependence of n-point functions could thus be calculated.

5.2.3 What one should show?

The ambitious goal is to show that $CH = \cup_m CH_{+-}(m)$, the notion of space-time as a 4-surface $X^4 \subset H$, and partonic picture (effective 2-dimensionality) summarized as $X^2 \subset X^4 \subset H$ emerges from the number theoretic sub-factor double induced by $C \subset H \subset O$ automatically. The notion of imbedding space and identification of space-time as surface of imbedding space would thus emerge from number theory alone.

1. The Cayley-Dickson construction giving a series of algebras obtained by adding the * operation of existing algebra as a new imaginary unit to existing algebra indeed defines a von Neumann algebra and one would have infinite series of inclusions. The resulting imaginary unit obviously does not commute with existing imaginary units so that the commuting hyper-octonionic imaginary unit would not result in this manner. One should show that the commuting imaginary unit emerges naturally.

2. Bott periodicity states that Clifford algebras are in well defined sense equivalent *mod* 8. Hence one can ask could the further continuation would give $C \subset H \subset O \subset O \subset ...$ at the level of von Neumann algebras also in accordance with the non-existence of further number fields.

3. More concretely, one should demonstrate that the configuration space spinor fields satisfying the Super Virasoro conditions form a family parameterized by hyper-quaternionic coordinate m appearing as an expansion parameter of field and mass squared formula characterizes the dependence on m. If one can start from the parametrization of CH spinors (as opposed to spinor fields) as Laurent series of hyper-quaternion m without any a priori space-time interpretation there are hopes for this.

   This is the case if configuration space spinors expressible as a Laurent series in hyper-quaternionic/-octonionic parameter having operator coefficients in H-/O-Clifford algebra are a natural notion. These series would corresponds to a generalization of corresponding expansions in complex coordinate for Kac-Moody and super-conformal algebra elements and conformal fields in general. This suggests the notion of local von Neumann algebras and their inclusions as a key concept. Note that this does not yet imply the notion of configuration space spinor field.

5.2.4 Clues

The construction of the representations of super canonical and super Moody algebras involves the choice of a fixed future or past light cone since super-canonical algebra is defined at its boundary. Since four momentum labels the states of these representations, also the translational degrees of freedom associated with the tip of the light cone brings in the desired moduli. If one can show that super-conformal symmetries accompany naturally the inclusions of von Neumann algebras, one could also understand the emergence of hyper-quaternionic M^4 coordinate as moduli. The allowed values of the index $\mathcal{M} : \mathcal{N}$ indeed label minimal conformal field theories and quantum groups associated with conformal field theories. Therefore there are reasons to optimism.

The hyper-quaternionic inverse fails to exist when the hyper-quaternion is light-like. M^4_{+-} would thus define converge region of the Laurent series, and explain the restriction of space-time surfaces inside $M^4_\pm$ (classical causality) implying light cone cosmology. This would bring in the desired super-canonical conformal invariance in $\delta M^4_+ \times CP_2$ and four-momentum and hyper-quaternionic parameter space would have the desired interpretation as M^4.

5.2.5 Two options for defining infinite-dimensional Clifford algebras

There are two options concerning the definition of infinite-dimensional Clifford algebra.

1. One could define them using matrix algebra $C(k)$, $k = 2,4,6,8$ as basic building brick. Bott periodicity implies that $C(8) \otimes \mathcal{M}$ and $\mathcal{M}$ are isomorphic as infinite-dimensional Clifford algebras.

2. The elements of matrix algebras belong naturally to some number field and in the matrix algebras $M_2(F)$, 2×2 matrices having $F = C, H$, and possibly even O valued matrix elements, could be considered as the building block of configuration space Clifford algebra.

The algebras defined by $M_2(C)$ and $M_2(H)$ certainly exists with latter having real coefficients as Abelian coefficient ring. $M_2(C)$ matrices can be also regarded as complexified quaternions H_C with Abelian complexification. Hence it seems that $M_2(H_C) = M_2(C)\otimes H = H_C\otimes H = M_2(C)\otimes M_2(C)$ as a building block gives subalgebra of the algebra generated by $M_2(C)$ having only $C(4)$ elements with coefficients in the ring defined by the infinite tensor product of 16×16 matrices.

The matrix algebra defined by $M_2(O_C) = M_2(C) \otimes O$ where C commutes with O is non-associative and one can question the idea that it could define elegantly an infinite-dimensional spinor algebra. Hence it would seem safest to consider four options: Cl_2 generated by $M_2(C)$ plus the algebras Cl_k generated by $C(k)$, $k = 4,6,8$. In TGD framework $Cl_8 \equiv Cl$ is natural since configuration space Clifford algebra can be naturally regarded as an infinite tensor power of Clifford algebra of the imbedding space and since space time spinors are induced from H-spinors.

5.2.6 The replacement of infinite-dimensional Clifford algebras with their local variants

Hyper-quaternions/-octonions should naturally relate to a local algebra defined by the von Neumann algebra analogous to Kac Moody, Virasoro, and conformal algebras. One can indeed consider local hyper-F variants for all algebras defined by the sequence of Jones inclusions.

There is however a very delicate point involved. The power series in hyper-F coordinate variable with Taylor coefficients in infinite-dimensional Clifford algebra Cl belong to the tensor product $F_C \otimes Cl$. The coordinate of hyper-F is representable as a linear combination of Clifford algebra generators of M^k, $k = 2,4,8$.

If these generators have matrix representations as gamma matrices, the resulting algebra can be absorbed to Cl (defined most naturally as an infinite tensor power of $C(8)$) as a finite-dimensional tensor factor so that no local gauge algebra results. Only for 8-dimensional case in which Clifford algebra generators can be represented as octonionic units situation changes since non-associativity does not allow this absorption. From the power series of hyper-octonion one obtains by a restriction to a maximal associative subspace power series in the quaternionic coordinate.

One can still wonder why just hyper-octonions and hyper-quaternions. The construction of the quantum variants of complexified quaternions and octonions by replacing the coefficients by non-Hermitian operators provides the answer to this question. Complexified M^D, call it M^D_C, can be represented as a space spanned by Clifford algebra generators and its quantum counterpart is obtained by replacing the coefficients with non-Hermitian operators. The points of the real Minkowski space are represented as eigenstates of the hermitian operators $m^k_C + (m^k_C)^{dagger}$. These coordinate operators indeed commute so that their spectra define ordinary M^8 and its sub-manfolds M^4 as genuine quantal concepts. For Euclidian sub-space of maximal dimension the commutativity fails. This point is discussed in detail in the section devoted to the quantization of Planck constant.

This picture leads to a generalization of Jones inclusion to the Jones inclusion of local von Neumann algebras. It would not be surprising if local von Neumann algebras could be regarded as direct integral of factors. One might hope that the local variants of number theoretical Clifford algebras could be regarded as maximal extensions of them analogous to local number fields.

Since configuration space gamma matrices associated with CH_+ generate the Clifford algebra, the generalization boils down to an extension of these gamma matrices to hyper-quaternion valued fields expressible as power expansion in hyper-quaterionic coordinate. These fields are very much analogous to the corresponding fields appearing in super-string model. The difference is that these fields are non-hermitian, do not satisfy Majorana property, and carry well defined lepton or quark numbers [B4]. Anticommutation relations for these fields should fix the anticommutation relations for the operator coefficients for powers of hyper-quaternion coordinate. The outcome is admittedly very stringy and TGD can be interpreted as a generalization of super string model.

Similar relations are obtained for gamma matriecs at the partonic boundary components corresponding and $C \subset H$ inclusion relates them to the gamma matrices for $X^3 \subset X^4(X^3)$ whereas $X^4 \subset H$ relates these gamma matrices to the quantized versions of H gamma matrices. As a matter fact, induction procedure for octonionic quantum gamma matrices of H should give the gamma matrices at hyper-quaternionic space-time surfaces and further induction procedure at hyper-complex sections of partonic orbits. Therefore also induction procedure fits nicely TGD view.

5.2.7 The emergence of space-time as a four-surface and $HO - H$ duality

HO-H duality [E2] states that space-time surface can be equivalently regarded as surface in hyper-octonionic imbedding space and $M^4 \times CP_2$. This duality can be understood in the proposed framework.

1. The Jones inclusion for the local Clifford algebra involves the restriction of hyper-O (-H) coordinate to hyper-H (-C) coordinate to guarantee associativity in calculation of S-matrix elements. The most general inclusion of this kind gives rise to a hyper-quaternionic sub-manifold X^4 of hyper-octonions HO.

2. The identification of X^4 as a 4-surface in $M^4 \times CP_2$ results from the local selection of the hyper-quaternionic Clifford algebra as subalgebra assigning to a point of X^4 also point of M^4 besides point of both M^4.

3. The notion of configuration space spinor field follows by allowing quantum superpositions of configuration space spinors $\Psi(X^4 \subset M^4_\pm(m) \times CP_2)$ over X^4. The super-canonical algebra associated with the light-cone boundary leads to the existing construction of super-canonical and Kac-Moody representations. Super-canonical representations can be assigned with $H \subset O$ inclusion and super Kac-Moody representations with $C \subset O$ inclusion.

4. It deserves to be noticed that the construction does not require introduction of more general structures than future and past light cones at the basic level. This simplifies dramatically the construction of configuration space geometry.

5.2.8 Configuration space gamma matrices as hyper-octonionic conformal fields having values in HFF?

The fantastic properties of HFFs of type II_1 inspire the idea that a localized hyper-octonionic version of Clifford algebra of configuration space might allow to see space-time, embedding space, and configuration space as emergent structures. Surprisingly, commutativity and associativity imply most of the speculative "must-be-true's" of quantum TGD.

Configuration space gamma matrices act only in vibrational degrees of freedom of 3-surface. One must also include center of mass degrees of freedom which appear as zero modes. The natural idea is that the resulting local gamma matrices define a local version of HFF of type II_1 as a generalization of conformal field of gamma matrices appearing super string models obtained by replacing complex numbers with hyper-octonions identified as a subspace of complexified octonions. As a matter fact, one can generalize octonions to quantum octonions for which quantum commutativity means restriction to a hyper-octonionic subspace of quantum octonions. Non-associativity is essential for obtaining something non-trivial: otherwise this algebra reduces to HFF of type II_1 since matrix algebra as a tensor factor would give an algebra isomorphic with the original one. The octonionic variant of conformal invariance fixes the dependence of local gamma matrix field on the coordinate of HO. The coefficients of Laurent expansion of this field must commute with octonions.

The world of classical worlds has been identified as a union of configuration spaces associated with $M^4_\pm$ labeled by points of H or equivalently HO. The choice of quantization axes certainly fixes a point of H (HO) as a point remaining fixed under $SO(1,3) \times U(2)$ ($SO(1,3) \times SO(4)$). The condition that hyper-quaternionic inverses of $M^4 \subset HO$ points exist suggest a restriction of arguments of the n-point function to the interior of $M^4_\pm$.

Associativity condition for the n-point functions forces to restrict the arguments to a hyper-quaternionic plane $HQ = M^4$ of HO. One can also consider the commutativity condition by requiring that arguments belong to a preferred commutative sub-space HC of HO. Fixing preferred real and imaginary units means a choice of $M^2 = HC$ interpreted as a partial choice of quantization axes. This has quite strong implications.

1. The hyper-quaternionic planes with a fixed choice of M^2 are labeled by points of CP_2. If the condition $M^2 \subset T^4$ characterizes the tangent planes of all points of $X^4 \subset HO$ it is possible to map $X^4 \subset HO$ to $X^4 \subset H$ so that $HO-H$ duality ("number theoretic compactification") emerges. $X^4 \subset H$ should correspond to a preferred extremal of Kähler action. The physical interpretation would be as a global fixing of the plane of non-physical polarizations in M^8: it is not quite clear whether this choice of polarization need not have direct counterpart for $X^4 \subset H$. Standard model symmetries emerge naturally. The resulting surface in $X^4 \subset H$ would be analogous to a warped plane in E^3. This new result suggests rather direct connection with super string models. In super string models one can choose the polarization plane freely and one expects also now that the generalized choice $M^2 \subset M^4 \subset M^8$ of polarization plane can be made freely without losing Poincare invariance with reasonable assumption about zero energy states.

2. One would like to fix local tangent planes T^4 of X^4 at 3-D light-like surfaces X^3_l fixing the preferred extremal of Kähler action defining the Bohr orbit. An additional direction t should be added to the tangent plane T^3 of X^3_l to give T^4. This might be achieved if t belongs to M^2 and perhaps corresponds to a light-like vector in M^2.

3. Assume that partonic 2-surfaces X belong to $\delta M^4_\pm \subset HO$ defining ends of the causal diamond. This is obviously an additional boundary condition. Hence the points of partonic 2-surfaces are associative and can appear as arguments of n-point functions. One thus finds an explanation for the special role of partonic 2-surfaces and a reason why for the role of light-cone boundary. Note that only the ends of lightlike 3-surfaces need intersect $M^4_\pm \subset HO$. A stronger condition is that the pre-images of light-like 3-surfaces in H belong to $M^4_\pm \subset HO$.

4. Commutativity condition is satisfied if the arguments of the n-point function belong to an intersection $X^2 \cap M^2 \subset HQ$ and this gives a discrete set of points as intersection of light-like radial geodesic and X^2 perhaps identifiable in terms of points in the intersection of number theoretic braids with $\delta H_\pm$. One should show that this set of points consists of rational or at most algebraic points. Here the possibility to choose X^2 to some degree could be essential. As a matter fact, any radial light ray from the tip of light-cone allows commutativity and one can consider the possibility of integrating over n-point functions with arguments at light ray to obtain maximal information. For the pre-images of light-like 3-surfaces commutativity would allow one-dimensional curves having interpretation as braid strands. These curves would be contained in plane M^2 and it is not clear whether a unique interpretation as braid strands is possible (how to tell whether the strand crossing another one is infinitesimally above or

below it?). The alternative assumption consistent with virtual parton interpretation is that light-like geodesics of X^3 are in question.

To sum up, this picture implies HO-H duality with a choice of a preferred imaginary unit fixing the plane of non-physical polarizations globally, standard model symmetries, and number theoretic braids. The introduction of hyper-octonions could be however criticized: could octonions and quaternions be enough after all? Could HO-H duality be replaced with O-H duality and be interpreted as the analog of Wick rotation? This would mean that quaternionic 4-surfaces in E^8 containing global polarization plane E^2 in their tangent spaces would be mapped by essentially by the same map to their counterparts in $M^4 \times CP_2$,and the time coordinate in E^8 would be identified as the real coordinate. Also light-cones in E^8 would make sense as the inverse images of $M^4_\pm$.

5.2.9 Quantal Brahman=Atman identity

The hierarchy of infinite primes (and of integers and rationals) [E3] was the first mathematical notion stimulated by TGD inspired theory of consciousness. The construction recipe is equivalent with a repeated second quantization of super-symmetric arithmetic quantum field theory with bosons and fermions labeled by primes such that the many particle states of previous level become the elementary particles of new level. The hierarchy of space-time sheets with many particle states of space-time sheet becoming elementary particles at the next level of hierarchy and also the hierarchy of n:th order logics are also possible correlates for this hierarchy. For instance, the description of proton as an elementary fermion would be in a well defined sense exact in TGD Universe.

This construction leads also to a number theoretic generalization of space-time point since given real number has infinitely rich number theoretical structure not visible at the level of the real norm of the number a due to the existence of real units expressible in terms of ratios of infinite integers. This number theoretical anatomy suggest kind of number theoretical Brahman=Atman principle stating that the set consisting of number theoretic variants of single point of the imbedding space (equivalent in real sense) is able to represent the points of the world of classical worlds or even quantum states of the Universe. Also a formulation in terms of number theoretic holography is possible.

Just for fun and to test these ideas one can consider a model for the representation of the configuration space spinor fields in terms of algebraic holography. I have considered guesses for this kind of map earlier [E10, E3] and it is interesting to find whether additional constraints coming from zero energy ontology and finite measurement resolution might give. The identification of quantum corrections as insertion of zero energy states in time scale below measurement resolution to positive or negative energy part of zero energy state and the identification of number theoretic braid as a space-time correlate for the finite measurement resolution give considerable additional constraints.

1. The fundamental representation space consists of wave functions in the Cartesian power U^8 of space U of real units associated with any point of H. That there are 8 real units rather than one is somewhat disturbing: this point will be discussed below. Real units are ratios of infinite integers having interpretation as positive and negative energy states of a super-symmetric arithmetic QFT at some level of hierarchy of second quantizations. Real units have vanishing net quantum numbers so that only zero energy states defining the basis for configuration space spinor fields should be mapped to them. In the general case quantum superpositions of these basis states should be mapped to the quantum superpositions of real units. The first guess is that real units represent a basis for configuration space spinor fields constructed by applying bosonic and fermionic generators of super-canonical and super Kac-Moody type algebras to the vacuum state.

2. What can one say about this map bringing in mind Gödel numbering? Each pair of bosonic and corresponding fermionic generator at the lowest level must be mapped to its own finite prime. If this map is specified, the map is fixed at the higher levels of the hierarchy. There exists an infinite number of this kind of correspondences. To achieve some uniqueness, one should have some natural ordering which one might hope to reflect real physics. The irreps of the (non-simple) Lie group involved can be ordered almost uniquely. For simple group this ordering would be with respect to the sum $N = N_F + N_{F,c}$ of the numbers N_F *resp.* $N_{F,c}$ of the fundamental representation *resp.* its conjugate appearing in the minimal tensor product giving the irrep. The generalization to non-simple case should use the sum of the integers N_i for different factors for factor groups. Groups themselves could be ordered by some criterion, say dimension. The states of a given representation could be mapped to subsequent finite primes in an order respecting some natural ordering of the states by the values of quantum numbers from negative to positive (say spin for $SU(2)$ and color isospin and hypercharge for $SU(3)$). This would require the ordering of the Cartesian factors of non-simple group, ordering of quantum numbers for each simple group, and ordering of values of each quantum number from positive to negative.

   The presence of conformal weights brings in an additional complication. One cannot use conformal as a primary orderer since the number of $SO(3) \times SU(3)$ irreps in the super-canonical sector is infinite. The requirement that the probabilities predicted by p-adic thermodynamics are rational numbers or equivalently that there is a length scale cutoff, implies a cutoff in conformal weight. The vision about M-matrix forces to conclude that different values of the total conformal weight n for the quantum state correspond to summands in a direct sum of HFFs. If so, the introduction of the conformal weight would mean for a given summand only the assignment n conformal weights to a given Lie-algebra generator. For each representation of the Lie group one would have n copies ordered with respect to the value of n and mapped to primes in this order.

3. Cognitive representations associated with the points in a subset, call it P, of the discrete intersection of p-adic and real space-time sheets, defining number theoretic braids, would be in question. Large number of partonic surfaces can be involved and only few of them need to contribute to P in the measurement resolution used. The fixing of P means measurement of N positions of H and each point carries fermion or anti-fermion numbers. A more general situation corresponds to plane wave type state obtained as superposition of these states. The condition of rationality or at least algebraicity means that discrete variants of plane waves are in question.

4. By the finiteness of the measurement resolution configuration space spinor field decomposes into a product of two parts or in more general case, to their superposition. The part Ψ_+, which is above measurement resolution, is representable using the information contained by P, coded by the product of second quantized induced spinor field at points of P, and provided by physical experiments. Configuration space "orbital" degrees of freedom should not contribute since these points are fixed in H.

5. The second part of the configuration space spinor field, call it Ψ_-, corresponds to the information below the measurement resolution and assignable with the complement of P and mappable to the structure of real units associated with the points of P. This part has vanishing net quantum numbers and is a superposition over the elements of the basis of CH spinor fields and mapped to a quantum superposition of real units. The representation of Ψ_- as a Schrödinger amplitude in the space of real units could be highly unique. Algebraic holography principle would state that the information below measurement resolution is mapped to a Schrödinger amplitude in space of real units associated with the points of P.

6. This would be also a representation for perceiver-external world duality. The correlation function in which P appears would code for the information appearing in M-matrix representing the laws of physics as seen by conscious entity about external world as an outsider. The quantum superposition of real units would represent the purely subjective information about the part of universe below measurement resolution.

There is an objection against this picture. One obtains an 8-plet of arithmetic zero energy states rather than one state only. What this strange 8-fold way could mean?

1. The crucial observation is that hyper-finite factor of type II_1 (HFF) creates states for which center of mass degrees of freedom of 3-surface in H are fixed. One should somehow generalize the operators creating local HFF states to fields in H, and an octonionic generalization of conformal field suggests itself. I have indeed proposed a quantum octonionic generalization of HFF extending to an HFF valued field Ψ in 8-D quantum octonionic space with the property that maximal quantum commutative sub-space corresponds to hyper-octonions. This construction raises $X^4 \subset M^8$ and by number theoretic compactification also $X^4 \subset H$ in a unique position since non-associativity of hyper-octonions does not allow to identify the algebra of HFF valued fields in M^8 with HFF itself.

2. The value of Ψ in the space of quantum octonions restricted to a maximal commutative subspace can be expressed in terms of 8 HFF valued coefficients of hyper-octonion units. By the hyper-octonionic generalization of conformal invariance all these 8 coefficients must represent zero energy HFF states. The restriction of Ψ to a given point of P would give a state, which has 8 HFF valued components and Brahman=Atman identity would map these components to U^8 associated with P. One might perhaps say that 8 zero energy states are needed in order to code the information about the H positions of points P. The condition that Ψ represents a state with vanishing quantum numbers gives additional constraints. The interpretation inspired by finite measurement resolution is that the coordinate h associated with Ψ corresponds to a zero energy insertion to a positive or negative energy state localizable to a causal diamond inside the upper or lower half of the causal diamond of observer. Below measurement resolution for imbedding space coordinates $\Psi(h)$ would correspond to a nonlocal operator creating a zero energy state. This would mean that Brahman=Atman would apply to the mini-worlds below the measurement resolution rather than to entire Universe but by algebraic fractality of HFFs this would would not be a dramatic loss.

5.3 Relation to other ideas

5.3.1 Category formed by Clifford algebras as a basic structure

A proper mathematical framework seems to be a category having as objects the number theoretical Clifford algebras listed below and probably many others while Jones inclusion would define the fundamental arrow.

One can imagine huge variety of natural inclusions of Clifford algebras in TGD framework.

1. Each space-time sheet can be regarded as 3-surface belonging to a single particle sector of configuration space and the corresponding Clifford algebra can be included to various Clifford algebras associated with collections of space-time sheets containing this space-time sheet as a topologically condensed space-time sheet. This inclusion is certainly a fundamental one. Universe as a computer idea would encourage to think that Universe is utilizing this kind of inclusions to mimic itself.

2. The $R_0^G \subset R^G$ type inclusions seem to be associated with non-perturbative phases with modified values of Planck constants and scaling factors of metrics in M^4 and CP_2 degrees of

freedom. Groups $G = G_a \times G_b \subset SL(2,C) \times SU(2)$ characterize these inclusions and modular subgroups of SL(2,Z) and their complexificiations which are unit matrices modulo p^k, are of special interest concerning p-adicization by algebraic continuation. The group theoretical discretization using an extension of rationals for F in $SL(2,F)$ is in accordance with the decomposition $CH = \cup_a CH_a$, a the dip of light cone.

3. Number theoretical inclusion sequence defines a canonical inclusion sequence. For this inclusion N is very small as compared to M, which suggest that the index is infinite. A concrete matrix representation would suggest an interpretation as an infinite tensor power of the standard $R_0^G \subset R^G$ inclusion so that the index would be an infinite power of $\mathcal{M} : \mathcal{N}$. For instance, $CL(O)$ would be $CL(H)_\infty$ in some sense. The index would be an infinite power of $M : N$ and remain finite only for $n = 3$ and corresponding groups A_2 (Z_2 and color group) and E_6 (tedrahedral group and E_6) would be special.

4. Also sub-algebras generated by $M_2(F)$ containing only matrices for which the ratios of elements belong to some algebraic extension of rationals are possible. Also this gives rise to inclusion hierarchies expected to be of special importance for p-adicization.

5.3.2 Dualities and number theoretic Jones inclusions

Dualities define a relatively new and speculative element in TGD. Jones inclusions provide also new insights and support to various proposed dualities of TGD.

1. 7-3 duality duality states roughly that TGD could be described either in terms of 3-dimensional light like causal determinants of space-time surfaces or in terms of the light cones $\delta M^4_+ \times CP_2$ or equivalently 7-D lightcones of HO. A concrete implication is the generalization of coset mechanism for superconformal invariance in which differences of super-canonical and super Kac-Moody generators annihilate physical states. A possible correlate for this duality would be the canonical pair of Jones inclusions $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{M} \subset \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$, which are dual.

   If Cl is generated by an infinite tensor power of $C(8)$, the inclusion sequence would be restricted to the tensor factor O_C in $O_C \otimes Cl$ and be represented as a sequence of $C_C \subset H_C \subset O_C$ of finite-dimensional inclusions. The quantum counterpart of C_C is quantum plane. In quantum case the identification of the index for this inclusion would be very naturally $\sqrt{\mathcal{M} : \mathcal{N}}$ reproducing correctly quantum variants of the dimensions in question.

2. HO-H duality follow naturally from the basic picture. In H description electro-weak quantum numbers are spinlike and color in CP_2 partial waves and HO description color is is spinlike and electro-weak quantum numbers correspond to E^4 partial waves.

3. Hyperquaterionic-co-hyperquaternionic duality follows from the possibility of identifying space-time surfaces as associative or co-associative sub-manifolds of HO. This duality means that space-time surfaces in HO become to certain extend a relative notion.

6 Construction of S-matrix and Jones inclusions

In this section following topics are discussed.

1. A general master formula for the construction of S-matrix is represented assuming that the generalization of duality symmetry of old-fashioned string models implying the reduction of diagrams to diagrams involving only single vertex makes sense. The S-matrix elements are obtained by replacing ordinary tensor product for free fields with Connes tensor product so that a hierarchy of S-matrices parameterized by Jones inclusions results.

2. The hierarchy of Jones inclusions is shown to lead to a hierarchy of S-matrices in which Feynman diagrams of previous level can be said to represent states of the next level.

6.1 Construction of S-matrix in terms of Connes tensor product

An explicit first principle formula for S-matrix has remained an unfulfilled dream although a lot of progress have been made: mention only the notion of generalized Feynman diagram involving a generalization of the duality symmetry [C7]. In the following it will be found the replacement of the ordinary tensor product with Connes tensor product for the states created by free fields allows a very elegant general formula for S-matrix. The dependence of the S-matrix on Jones inclusion raises the interaction of observer with measured system in a central role. This dependence is already present in standard quantum field theory via the dependence of the S-matrix on ultraviolet and infrared cutoffs.

6.2 The challenge

The construction of S-matrix for a single space-time surface was discussed already in [C7] by introducing the notion of generalized Feynman diagram. Although this treatment seems to apply only to what happens inside single 3-D external line of Feynman diagrams the general picture is rather near to what seems to be correct one.

1. Lines correspond to 3-D light-like causal determinants (CDs) X^3_i at four-dimensional space-time sheets representing incoming particles. They can correspond to boundary components of a space-time sheet (such as boundaries of magnetic flux tubes) but can also serve as horizons separating maximal non-deterministic regions within a space-time sheet.

2. Vertices correspond to 2-D partonic surfaces at which the light-like CDs X^3_i branch like a lines of Feynman diagrams. X^3_i and also space-time surfaces would be singular as manifolds but only apparently as will be found. The 2-surfaces representing vertices need not have any singularities, say pinch like singularities appearing in stringy diagrams.

3. There is rather close analogy with the branes in the sense that the intersection of space-time surfaces 7-D light-like CDs $X^7_\pm = \delta M^4_\pm \times CP_2$ of imbedding space provide a natural gauge fixing for 4-D general coordinate invariance. Hence the incoming partons X^2_i correspond to intersections $X^3_i \cap X^7_\pm$. Future (past) oriented light-cone corresponds to incoming (outgoing) particles.

In this framework the basic challenges would be following.

1. Construct explicitly the unitary evolution operators associated with the lines possibly defining the analogs of propagators. These operators should be fixed to a high degree by the dynamics of the second quantized induced spinor fields as has been suggested in [C7]. The existence of a universal unitary von Neumann algebra automorphism Δ^{it} suggesting itself as a candidate for this unitary evolution operator. The identifiability of this automorphism as that defined by the modified Dirac operator is also suggestive.

2. Δ^{it} represents scaling operation is a mere inner automorphism for factors of type II_1, which suggests that both internal and external lines represent on mass shell particles in the sense that Virasoro conditions hold true and the automorphism represents braiding S-matrix. This in turn inspires the hypothesis that the Feynmann graphs can have only on mass shell particles as internal lines: by unitarity the S-matrix elements reduce to diagrams having only single vertex.

3. Vertices are in principle fixed as vacuum expectation values for the product of operators creating the incoming and outgoing states at the vertices. Connes tensor product suggests itself strongly. These operators are constructible from oscillator operators associated with the generalized eigen modes of the modified Dirac operator acting on the second quantized induced spinor fields, whose quantization is fixed by the requirement that the super-canonical charges constructed in terms of oscillator operators define configuration space gamma matrices having super-symmetrized symplectic transformations of $\delta M^4_+ \times CP_2$ as isometries. Intuitively it looks obvious that the product of the operators creating states at the lines defines vertex as its vacuum expectation. The challenge is to imbed the fermionic oscillator operator algebras associated with incoming lines to same oscillator algebra.

6.2.1 Master formula for S-matrix

The possibility to interpret configuration space spinor Clifford algebra elements as analogous of conformal fields in M^4 suggests that the notion of n-point function could serve as a useful starting point in the attempts to understand the general structure of S-matrix.

1. The general formula should reproduce generalized Feynman diagrams for which lines are space-time sheets whose ends meet at vertices which are 3-surfaces. The lines should correspond to the solutions of field equations having interpretation as generalized Bohr orbits so that classical theory should be an exact part of the construction of S-matrix.

2. The generalization of the duality symmetry of the old fashioned string model requires that all Feynman diagrams should be equivalent with a diagram involving single vertex from which all incoming and outgoing lines emanate. This picture is analogous to the description of scattering matrix elements in terms of effective action so that each connected n-point function corresponds to a diagram with a single vertex. This picture would suggest that one should not start the construction from incoming and outgoing particles and continue adding all possible collections of vertices between them as in perturbative quantum field theory. Rather, one should do just the reverse by starting from the vertex and gluing to it space-time sheets leading to the initial states at the boundaries of light cones assignable to the arguments of n-point function.

General coordinate invariance has turned out to be extremely powerful guiding principle in the construction of TGD and comes in rescue also now.

1. The reduction to a diagram which single vertex suggests that particle reactions are essentially processes of creation of particles from vacuum which propagate classically to the boundaries of light cones associated with the arguments of n-point function. Hence S-matrix elements would be basically expressible as amplitudes for creating from vacuum a 3-surface X^3 at which one can assign a product of elements of configuration space Clifford algebra creating from the vacuum state with well-defined fermionic and other quantum numbers. A creation of $N_{in} + N_{out}$ particles from vacuum is in question.

2. The resulting N_{out} positive energy (N_{in} negative energy) particles travel to the boundaries of future (past) light cones associated with points m_i appearing in the n-point function. General coordinate invariance implies that the values of configuration space spinor fields at the two ends of a given line are identical and thus expressible using their values at the corresponding light cone boundaries. This means enormous simplification since same basis of configuration space spinor fields in vibrational degrees of freedom can be used for all light cones involved and only the dependence on the coordinate characterizing the position of light cone complicates the situation. Lines would indeed be 4-surfaces, whose ends meet at the vertex. Crossing symmetry would be an automatic consequence in this picture.

The rules would be formally rather simple for the construction of n-point function with given points of M^4.

1. Assign to each 3-surface X^3 (possibly consisting of disjoint components) incoming and outgoing space-time surfaces ending at the boundaries of the light-cones involved and satisfying classical field equations guaranteing generalized Bohr orbit property.

2. Form the Connes tensor product of Clifford algebra elements associated with corresponding light cones and thus depending on their positions m_i, and calculate its vacuum expectation. The dependence on m_i and the possibility of vertices with arbitrarily high number of incoming and outgoing lines trivially guarantee that S-matrix is non-trivial.

3. The tensor product appearing in the vertex would be Connes tensor product $\otimes_N$ and vacuum expectation would be defined as the manifestly finite trace. The factors in the tensor product would be infinite-dimensional Clifford algebras M_i associated with the lines of the graph and $\mathcal{N}$ could be identified by the condition that $\mathcal{M}_\rangle/\mathcal{N}$ is the quantum variant of the Clifford algebra of H. Also the quantum variants of S-matrix corresponding to various groups $G = G_a \times G_b \subset SL(2,C) \times SU(2)$, $SU(2) \subset SU(3)$ would result from the same formula.

4. Perform ordinary functional integral over 3-surfaces X^3 defined by the exponent of Kähler function, which by the non-locality of the Kähler function as a functional of 3-surface and by the Ricci flatness of the configuration space geometry should be free of divergences. There are good reasons to hope that the radiative corrections to this integral sum up to zero around maxima of Kähler function. Super-symmetry raises the hope that also configuration space degrees of freedom correspond to a hyper-finite type II_1 factor and could be treated very much like fermionic degrees of freedom.

This description is not the only one can imagine. More complex tree diagrams with propagator lines would be obtained by allowing several vertices connected by internal lines represented by space-time sheets. The generalization of the duality symmetry says that these diagrams provide an alternative description equivalent with the minimal one. The classical non-determinism of Kähler action indeed forces to consider this possibility (recall that for CP_2 type extremals representing elementary particles the M^4 projection is light-like random curve which implies classical Virasoro conditions).

6.2.2 How to understand the unitarity of S-matrix

It should be possible to understand the unitarity of S-matrix in a simple manner from the proposed master formula for the S-matrix.

1. Connes tensor product is responsible for the non-triviality of S-matrix

The basic observation is that the presence of M^4 coordinates dependence makes configuration space gamma matrices analogous to quantum fields. Gamma matrices represent free fields as in string models and conformal field theories. S-matrix is obtained using ordinary inner product and by replacing the ordinary tensor product with Connes tensor product. Thus Connes tensor product would represent interactions and would be essential for the non-triviality of S-matrix. The beauty of this picture is that the quantum number spectrum of free field theory is preserved as such and there is no need to introduce the notion of virtual states or interaction terms.

Connes tensor product reduces degrees of freedom from the ordinary tensor product and there is a close analogy with the dynamics of space-time surfaces. Imbedding space metric, gamma matrices, and gauge fields are non-dynamical but their projection to the space-time surface makes them dynamical. Gamma matrices are free gamma matrix fields with trivial S-matrix but the restrictions posed by the Connes tensor make the dynamics non-trivial. Connes tensor product thus represents something analogous to the restriction of Clifford algebra to its sub-algebra.

2. Unitary S-matrix as a representation of crossing operation

In the proposed picture states with positive and negative energies are created from vacuum. By expressing S-matrix elements as amplitudes for the generation of a state containing incoming particles as positive energy particles and outgoing particles as negative energy particles, unitarity condition can be transformed by crossing symmetry to orthogonality condition of negative (positive) energy states assuming their completeness. Therefore S-matrix represents unitary crossing operation transforming positive energy bra to negative energy ket. Unitarity states that crossing operation and its conjugate produce a trivial outcome.

3. Connes tensor product as a representation for the unitary crossing operation

Vacuum expectation of the Connes tensor product of localized gamma matrices creating a zero energy states would be equal to the inner product of outgoing and incoming states created by the ordinary tensor products of the same gamma matrices. Connes tensor product should guarantee unitarity of S-matrix.

Gamma matrices would behave as free fields with respect to the ordinary tensor product and S-matrix would be trivial. Super Virasoro conditions would give the mass spectrum. $\mathcal{N}$ reducing to a unit matrix would define a trivial S-matrix. Free field property is essential for the finiteness of the theory since Connes tensor product and finite trace cannot induce infinities.

Connes tensor product would make the S-matrix non-trivial. Any $\mathcal{N}$ would define non-trivial S-matrix via Connes tensor product and a hierarchy of S-matrices would result. In what sense the S-matrix is unitary and whether it is so is of course not obvious. Since $\mathcal{N}$ takes the role analogous to complex coefficient field in quantum mechanics one is forced to ask whether a reduction of single particle degrees of freedom to $\mathcal{M}_k/\mathcal{N}$ occurs so that unitarity holds true in these degrees of freedom. In what sense it holds true is not quite obvious. The mathematical challenge is to prove that Connes tensor product indeed gives rise to a unitary S-matrix in the proposed framework.

6.2.3 Jones inclusion as a representation of quantum measurement

The number of observable degrees of freedom is finite in any experiment. Since the number of degrees of freedom for the particle is infinite, the experimental situation must somehow leave almost all of these degrees of freedom undetected. $\mathcal{N} \subset \mathcal{M}_k$ must represent the interaction of the observer with the measured system. Finite-fractal dimensional $\mathcal{N}$ module $\mathcal{M}_k/\mathcal{N}$ would represent those gamma matrices which define the observable degrees of freedom.

There are two interpretations for this.

1. $\mathcal{N}$ represents those degrees of freedom in which there are no correlations between measurement system and measured system and experimenter entangles with $\mathcal{M}_k/\mathcal{N}$ degrees of freedom. The division by $\mathcal{N}$ could be also interpreted as being due to a finite measurement accuracy implying a thinning of those degrees of freedom in which the state function reduction can occur. It is not clear whether $\mathcal{N}$ must be same for all particles and one might argue that this need not be the case. The entanglement leading to state function would occur only in finite number of degrees of freedom characterized by $\mathcal{M}_k/\mathcal{N}$.

2. A completely opposite interpretation is that $\mathcal{N}$ represents Clifford subalgebra shared by particles and observed with the property that entanglement in this degrees of freedom is stable in the time scale of the experiment. This would leave only $\mathcal{M}_k/\mathcal{N}$ degrees of freedom as those for which state function reduction occurs in the time scale of the scattering experiment. This option would explain naturally why $\mathcal{N}$ is same for all particles.

3. Also the combination of above views is possible. Degrees of freedom in which dynamics is very rapid *resp.* slow would correspond to the case 1) *resp.* 2).

Some further remarks are in order.

1. The fractal dimension $\mathcal{M} : \mathcal{N}$ tells that the correlations due to non-commutativity reduce their effective number.

2. If all degrees of freedom could be measured $\mathcal{N}$ would reduce to an algebra containing only unit and S-matrix would become trivial. The non-triviality of S-matrix would is thus due to the interaction between the experimenter and the system studied. In quantum field theories length and time scale cutoffs would represent this fact in a rough manner.

3. Γ matrices are not Hermitian since they are essentially superpositions of fermionic oscillator operators. Fourier transforms of Gamma matrices could be used to define occupation number operators in the momentum space as natural observables.

6.2.4 Precise definition of the notion of unitarity for Connes tensor product

The previous physical picture helps to characterize the notion of unitarity precisely for the S-matrix defined by Connes tensor product. For simplicity restrict the consideration to configuration space spin degrees of freedom.

1. $Tr(Id) = 1$ condition implies that it is not possible to define S-matrix in the usual sense since the probabilities for individual scattering events would vanish. Connes tensor product means that in quantum measurement particles are described using finite-dimensional quantum state spaces $\mathcal{M}/\mathcal{N}$ defined by the inclusion. For standard inclusions they would correspond to single Clifford algebra factor $C(8)$. This integration over the unobserved degrees of freedom is nothing but the analog for the transitions from super-string model to effective field theory description and defines the TGD counterpart for the renormalization process.

2. The intuitive mathematical interpretation of the Connes tensor product is that $\mathcal{N}$ takes the role of the coefficient field of the state space instead of complex numbers. Therefore S-matrix must be replaced with $\mathcal{N}$-valued S-matrix in the tensor product of finite-dimensional state spaces. The notion of $\mathcal{N}$ unitarity makes sense since matrix inversion is defined as $S_{ij} \rightarrow S^{\dagger}_{ji}$ and does not require division (note that i and j label states of $\mathcal{M}/\mathcal{N}$).

3. The probabilities P_{ij} for the general transitions would be given by

$$P_{ij} = N_{ij}N^{\dagger}_{ij} \ , \quad (9)$$

and are in general $\mathcal{N}$-valued unless one requires

$$P_{ij} = p_{ij}e_{\mathcal{N}} \ , \quad (10)$$

where $e_{\mathcal{N}}$ is projector to $\mathcal{N}$. N_{ij} is therefore proportional to $\mathcal{N}$-unitary matrix. S-matrix is trivial in $\mathcal{N}$ degrees of freedom which conforms with the interpretation that $\mathcal{N}$ degrees of freedom remain entangled in the scattering process (option b)).

4. If S-matrix is non-trivial in $\mathcal{N}$ degrees of freedom, these degrees of freedom must be treated statistically by summing over probabilities for the initial states. The only mathematical expression that one can imagine for the scattering probabilities is given by

$$p_{ij} = Tr(N_{ij}N_{ij}^{\dagger})_{\mathcal{N}} \ . \tag{11}$$

The trace over $\mathcal{N}$ degrees of freedom means that one has probability distribution for the initial states in $\mathcal{N}$ degrees of freedom such that each state appears with the same probability which indeed was von Neumann's guiding idea. By the conservation of energy and momentum in the scattering this assumption reduces to the basic assumption of thermodynamics.

5. An interesting question is whether also momentum degrees of freedom should be treated as a factor of type II_1 although they do not correspond directly to configuration space spin degrees of freedom. This would allow to get rid of mathematically unattractive squares of delta functions in the scattering probabilities.

6.2.5 Conformal invariance and field theory and stringy phases

$\mathcal{M} : \mathcal{N} < 4$ assigns a unique minimal conformal field theory to the inclusion and this should give important information about the vertex. A priori the inclusions $\mathcal{N} \subset \mathcal{M}_k$ can have different values of $\mathcal{M}_k : \mathcal{N}$ determining the quantum phases q_i. Both physical intuition and anyonic statistics encourage to think that the values of quantum phases q_i are identical.

It seems conceivable that $\mathcal{M} : \mathcal{N} < 4$ vertices correspond physically to the low energy phase symmetry broken phase possible describable using renormalized field theory. Ordinary QFT would corresponds to $\mathcal{M} : \mathcal{N} \to 4$ limit whereas $\mathcal{M} : \mathcal{N} = 4$ phase with Kac-Moody symmetry would correspond to the "stringy" phase of the theory. The low energy limit would transform from an approximate theoretical description to an actual physical phase. In this phase massivation of massless particles would occur by p-adic thermodynamics [F2] whereas ultra-heavy particles would drop from the spectrum.

Dimensional regularization with complex space-time dimension $D = 4 - \epsilon \to 4$ could be interpreted as the limit $\mathcal{M} : \mathcal{N} \to 4$. $\mathcal{M}$ as an $\mathcal{M} : \mathcal{N}$-dimensional $\mathcal{N}$-module would provide a concrete model for the a quantum Clifford algebra. An entire sequence of counterparts of regularized theories corresponding to the allowed values of $\mathcal{M} : \mathcal{N}$ is predicted.

As will be discussed, the evolution of Jones index could correspond to renormalization group evolution for phase resolution characterized by the value $\hbar$.

In $\mathcal{M} : \mathcal{N} = 4$ case all ADE type $k = 1$ conformal theories are in principle possible. An open question is whether actually the conformal field theory defined by the Kac-Moody algebra characterizing TGD is possible or whether the idea about interfaces as able to emulate any string model is more appropriate.

6.2.6 Braiding and S-matrix

The S-matrices associated with braiding were the inspiration leading to the new view about Feynman diagrams which I have attempted to formulate in [C7] in terms of bi-algebras. The trace of S-matrix associated with braid defines knot invariant. Each compact Lie group gives to its own S-matrix and one can assign to each strand of the braid its own representation of the group. Same applies to knotted links in 3-space to which one can assign the trace of non-integrable phase factor. Functional integral average using 3-dimensional Chern-Simons action defines the topological quantum field theory allowing to calculate the associated invariants.

1. *Braiding can appear in two manners*

The natural expectation is that these braiding matrices emerge in the proposed framework. A natural looking idea is that the incoming and outgoing lines parton lines represented by light like causal determinants could carry out these braids. The closed partonic 2-surfaces would define the 2-spaces carrying the anyons. This interpretation would be very natural since anyonic statistics is indeed possible only in 2-dimensional context. This would require the generalization of the proposed formula: it would not be possible to get completely rid of external lines since unitary S-matrices representing braiding should be associated with every lightlike causal determinant. Whether these unitary matrices have interpretation in terms of Connes tensor product is an open question.

Braiding could emerge also in a second manner. The 3-surface representing the N-vertex over which the functional integral is performed could also represent the braiding. It would seem natural to assume that these 3-surfaces are space-like. For instance, the boundary component of the 3-surface could be constructed from that having spherical topology by building handles as threadlike wormholes (think of an apple!) connecting punctures at the boundary of spherical boundary component. These wormholes could get linked and knotted would define the braiding naturally. In these cases the physical states would represent braiding S-matrix as entanglement coefficients between states localized to the punctures would represent the S-matrix. This S-matrix would depend on the state since the representation of the gauge group G could be chosen freely for each puncture and the choices could be different for initial and final states. The contractions of oscillation operators in Connes tensor product would give quantum traces of $Tr_q(S_1S_2^\dagger)$ in the S-matrix element and thus the braid invariants would appear in S-matrix.

2. How the two kinds of braidings and Jones inclusions relate?

Braid S-matrix emerges in topological quantum field theory defined by Chern-Simons action. Only topological degrees of freedom and the moduli space of flat connections defines the genuine dynamical variables. The obvious questions relate to the interpretation of the braid S-matrix in TGD: is it associated with either space-like or lightlike braidings and is it associated with Jones inclusions labelled by finite subgroups $G \subset SU(2)$ or with Jones inclusions with $G = SU(2)$?

1. Quantum traces appears in the invariant would which would suggest that for $q < 1$ the theory should be assigned to the Jones inclusion $R_0^G \subset R^G$. If so q would actually correspond to the quantum counterpart of the subgroup of $SU(2)$ defining a minimal conformal field theory. ADE correspondence would however assign with it gauge group $\hat{G}$ and the proposed construction of multiplets of gauge group would provide the correspondence with flesh and bones. The ADE diagram for $SU(2)$ is not allowed for Jones inclusions. $SU(2)$ is however the minimal $n = 5$ option allowing universal topological quantum computation using braids. Hence this option would be more naturally associated with the braids assignable to space-like 3-surface defining the vertex and would give rise to various topological invariants.

2. The inclusions could also correspond to $q = 1$ and $G = SU(2)$. All simply laced ADE groups are possible label the inclusion. The theory would be conformal field theory with Kac-Moody symmetry assignable to ADE group $\hat{G}_b$. As already explained, also now subgroups of $SU(2)$ could appear naturally in TGD framework but $G_b \subset SU(2) \subset SU(3)$ would collect points of geodesic sphere S^2 of CP_2 to multiplets allowing to represent the group algebra of G_b allowing to realize the representations of corresponding Lie group $\hat{G}_b$ defining the Kac-Moody group. Also $\hat{G}_b = SU(2)$ would be possible and braiding would be realized at the geodesic sphere of CP_2. Also in this case the subgroup G would assign quantum group parameter q naturally to the conformal field theory although it is not associated with Jones inclusion directly.

The simplest realization would be in terms of cosmic strings $X^2 \times S^2$, with S^2 perhaps depending on point of X^2. For ordinary cosmic strings the enormous string tension and would make this option energetically impossible. The situation is not change by the change of values of $\hbar$ since mass squared operators are invariant under the scaling of Planck constants associated with M^4 and CP_2 degrees of freedom.

This realization could be naturally assigned to the light-like causal determinants associated with the external lines to which ordinary Kac Moody symmetry is naturally associated in TGD framework. The space-time surface reduces to $X^1 \times S^2$ at the light-like causal determinants with X^1 a light like geodesic of M^4. The ends of the bosonic strings indeed move with light velocity. In the interior of X^4 there is no need to required $X^2 \times S^2$ decomposition which looks too strong a condition. Note that S^2 would represent topological magnetic monopole appearing in the model of high T_c superconductivity in TGD Universe [J1, J2, J3].

The identification of the gauge group associated with light-like causal determinants is naturally based on standard model gauge symmetries. Interestingly, the outcome of p-adic mass calculations depends on the number of tensor factors of Super Virasoro representation only with no dependence on what the actual Kac Moody groups associated with the factors are.

3. Holonomy of $\mathcal{N} \subset \mathcal{M}$ and unitarity of S-matrix and braid statistics

The reduction of Δ^{it} to inner automorphism for type II_1 factors need not mean that they would not have any role in the theory. The first thing coming in mind would be that the modular S-matrices assignable to the braids assignable to external lines and the 3-surface defining vertex could reduce to Δ^{it} for some value of parameter t.

Assume that $\mathcal{N}$ is imbedded into each factor $\mathcal{M}_k$ appearing in external line of the S-matrix. The automorphisms induced by $\Delta_{\mathcal{M}_k}$ assignable to the strand of the braid defines a closed path of $\mathcal{N}$ in $\mathcal{M}$. The action on the automorphism on individual points of $\mathcal{N}$ could be non-trivial and would have interpretation as a holonomy group defining a unitary action of $\mathcal{M}$ on $\mathcal{N}$.

Single particle S-matrix would represent this action. This action could relate to the 2-dimensional braid statistics defined by quantum group. For braids the 2π braid rotation of $(k+1)^{th}$ strand around k^{th} strand induces a non-trivial action on the state and this action could correspond to $\Delta_{\mathcal{M}_k}$ holonomy on $\mathcal{N}$. This interpretation would predict that the S-matrix elements for diagrams differing by braidings of partons inside incoming and outgoing lines (3-D light-like causal determinants) are not identical.

6.3 What the equivalence of loop diagrams with tree diagrams means?

The generalization of the duality of old-fashioned string models leads, not only to the equivalence of loop diagrams with tree diagrams but to the equivalence with diagrams involving only single N-vertex. This outcome leads then to the master formula of S-matrix in terms of Connes tensor product in which the original principle does not anymore seem to play any role. The question is whether the generalization of duality should be given up as obsolete or whether it has some non-trivial meaning as believed originally [C7].

6.3.1 Cancellation of loop corrections for Feynman diagrams

One can ask whether the cancellation loop corrections for ordinary Feynman diagrams in some sense could provide an alternative to state the proposed equivalence. The triviality of the automorphism Δ^{it} for hyperfinite factors of type II_1 serving as a universal candidate for an object defining a propagator has a natural interpretation as an on mass shell property of all internal lines of ordinary Feynman diagrams with arbitrarily high N-particle vertices allowed.

One can indeed allow all possible diagrams if this condition is posed. The sum of all intermediate states resulting in the decay of the particles of internal lines to on mass shell particles and their scattering to the original state yields just $SS^\dagger = 1$ factor on internal line or set of lines and produces unity. Hence one can say that $i(T - T^\dagger) - TT^\dagger$ represents on mass shell loop corrections which vanishes by unitarity. This interpretation is the only possible one in the proposed framework.

6.3.2 The equivalence of loop diagrams with tree diagrams for generalized braid diagrams

The interpretation is that a diagram with loop is equivalent with diagram with no loop. Diagram in this sense cannot correspond to ordinary Feynman diagram, in particular not the Feynman diagram with only on mass shell loops. Conditions for the equivalence in this sense have been formulated in algebraic terms for the generalization of ribbon algebras [C7]. The foregoing argument leaves the possibility to assign the equivalence of loop diagrams with tree diagrams to generalized braid diagrams so that also in this case diagrams could be reduced to a diagram with single vertex. If Universe is mimicking itself by using generalized braid S-matrix to emulate the proper S-matrix this equivalence would provide a representations for the cancellations of loop corrections for proper S-matrix.

1. What the equivalence of loop diagrams with tree diagrams could mean

Since the orbits of light like causal determinants are determined by field equations, only the failure of classical determinism can allow diagrams with loops or more complex diagrams involving what would be interpreted as particle creation in string model context. The equivalence would state that non-determinism has interpretation as a kind of gauge symmetry. In the case of 3-surfaces appearing as vertices this interpretation would state that the S-matrix assignable to the space-like braid is same for all 3-surfaces obtained from each other by this equivalence. All the considerations of [C7] would relate to these diagrams.

The stringy loops in which partonic 2-surface decays temporarily to two partonic 2-surfaces do not correspond in TGD framework to particle decays but to a single particle propagation along two different paths simultaneously. This picture leads to a generalization of the quantum measurement theory and explanation [K1] for the findings of Shahriar Afshar relating to double slit experiment challenging Copenhagen interpretation [54]. For instance, in double slit experiment the measurement of the particle aspect of photon would reduce the branched photon path in such a manner that second branch corresponds to a vacuum extremal having a vacuum line representing identity operator as its algebraic counterpart. In this case the equivalence of loop diagrams with tree diagrams looks obvious.

The equivalence of loop diagrams with tree diagrams in this sense means that one can move the end of any internal line until it becomes a tadpole loop which must represent vacuum line and can be eliminated. This means that all diagrams are equivalent to a simplicial complex representing the homology of planar disk D^2 with the ends of the external lines at the boundary circle of the disk and having Euler characteristic $E + F - L = -2$.

The equivalence implies also that any tree diagram containing N-vertices with arbitrary values of N can be transformed to a single M-vertex, where M is the number of incoming lines (for convenience al lines are regarded as incoming). The diagram can be also transformed to a diagram containing only 3-vertices. Number theoretic vision about the role of classical division algebras in TGD [E2] suggests that this symmetry is closely related to the octonionic triality reflecting itself also as the existence of 3 8-dimensional representations of $SO(8)$.

2. Equivalence of loop diagrams with tree diagrams and Jones inclusions

Consider now how the equivalence could be understood in terms of Jones inclusions. The argument below is a simplification of the algebraic conditions formulated in [C7] guaranteing also the possibility to move the ends of the lines around the graph.

1. If each incoming line is thought of as being imbedded in the same manner to a II_1 factor $\mathcal{M}$

then each incoming line of the vertex can be characterized by the same value of $\mathcal{M} : \mathcal{N}$, and one can assign to each line emanating from a vertex a representation of the same quantum group or Kac Moody group. The notions of product, co-product, and bi-algebra are well-defined [C7].

2. Assume that it is possible to transform the diagram to a diagram containing only 3-vertices by moving around the ends of the lines: this possibility should relate closely to octonionic triality [E2] underlying the vertex construction. As a consequence, all loops reduce to self-energy loops. Assume that the operator in the third line of the vertex is product of the operators associated with other two lines and the operator associated with two lines is a co-product of the operator in the third line. Under this assumption products and co-products in the self energy loops compensate each other and they are trivial.

3. Does the equivalence with tree diagrams imply unitarity of the generalized braid S-matrix?

It would be easier to take seriously the reducibility of generalized Feynman diagrams to tree diagrams if it would guarantee the unitarity of the generalized braid S-matrix. The following heuristics indicates that this could be the case.

1. The equivalence with tree diagrams allows to carry out two operations for the generalized Feynman diagrams.
i) It is possible to transform diagrams representing S-matrix elements to diagrams involving single vertex with M incoming lines and N outgoing lines. Incoming lines start from the boundary of a future directed light cone $X^7_+ = \delta M^4_+ CP_2$ and outgoing lines end at the boundary of a past directed light-cone $X^7_- = \delta M^4_+ \times CP_2$ having its tip inside X^7_+.
ii) It is possible to move the position of $M+N$ vertex arbitrarily near to the initial moment. At space-time level this means that the M partonic orbits intersect at the partonic 2-surface already at X^7_+ and decay to N partonic 2-surfaces.

2. Unitarity conditions should reduce to the statement that the initial states M_1 and M_2 are orthogonal. The sum over intermediate states in the unitarity relation involves a sum over number N of outgoing lines. Assume that it can be transformed by the completeness of states to a form in which a delta function appears stating that the values t_i are equal for N lines and their conjugates. If this is the possible, the unitary automorphisms Δ^{it_i} and their conjugates compensate each other for each outgoing N line.

3. If only the condition i) is assumed, the unitarity condition reduces to a condition stating the orthogonality of the images of the states $\hat{M}_1$ and $\hat{M}_2$ obtained from M_1 and M_2 by assigning the S-matrices S_i to the lines M_1 and their conjugates $S^\dagger_j$ to the lines of M_2. A reaction in which both incoming and outgoing partons belong to the boundary of the same future light-cone X^7_+ is in question. M particles travel to future, react in $M+N$ vertex and produce N particles, which are reflected back to the past.

4. If also the Feynman diagrams characterizing the S-matrix obtained by replacing the automorphisms associated with outgoing lines with their time reversals satisfy the equivalence with tree diagrams, this S-matrix is trivial without further conditions since the lines to future and back can be contracted to points. If also the condition ii) is assumed, $\hat{M}_i = M_i$ and unitarity conditions reduce to the ordinary orthogonality conditions.

6.4 Can one imagine alternative approaches?

The original approach to the construction of S-matrix was based on the idea that von Neumann algebras allow unique outer automorphism Δ^{it} which could define free propagation of particles whereas vertices would be defined using Connes tensor product as already discussed. This approach does not however work unless type III_1 factors are in question.

6.4.1 Of mass shell states are not possible for factors of type II_1

The original proposal for S-matrix was based on the observation that von Neumann algebras allow a universal unitary automorphism $A \rightarrow \Delta^{it} A \Delta^{-it}$ [20], which is unique apart from an inner automorphisms $\Delta^{it} \rightarrow U\Delta^{it}V$, with V U and V defining a change of basis for the target and domain. This automorphism looks a highly attractive candidate for the unitary transformation associated with the lines of generalized Feynman diagrams.

It came as a surprise, that this outer automorphism is trivial for factors of type I and II. In terms of Feynman diagrammatics the only conclusion is that the internal lines can be only on mass shell particles so that graphs must reduce to single vertex graphs as indeed implied by the generalized duality. The fact that the automorphism represents scaling would mean that on mass shell property means that Super Virasoro conditions are satisfied.

This picture is of course consistent with quantum classical correspondence and the absence of path integral. Super-symmetry suggests that also configuration space degrees of freedom can be treated in the similar manner. If so, the extension of S-matrix elements to p-adic number fields reduces to the extension of the traces of Connes tensor products to other number fields and their algebraic extensions. Even the S-matrices for p-adic–real transitions might be calculable in terms of n-point functions of conformal field theories with arguments restricted to the intersections of real and p-adic space-time sheets consisting of imbedding space points belonging to the algebraic extensions of rationals. Also the S-matrix hierarchy labelled by quantum groups would appear naturally.

6.4.2 Could local Clifford algebras give rise to III_1 factor?

The previous picture is extremely elegant and conforms with the basic philosophy of TGD. The localization of CH Clifford algebra with respect to M^4 and M^8 coordinates is however analogous to the replacement of gamma matrices with quantum fields defined in M^4. Since quantum field theory in M^4 leads to factors of type III_1, one must face the possibility that one after all has the situation in which Δ^{it} is non-trivial and defines the propagator as a universal automorphism. Even if this were the case, the generalization of the duality symmetry would leave only the diagrams containing only external lines so that Δ^{it} would appear nowhere.

If duality symmetry does not hold true, the analog of the ordinary stringy perturbation theory with Δ^{it} representing scaling would emerge and propagator would be essentially $1/(L_0 + i\epsilon)$. Vertices could be still defined as in previous case.

The first problem is how to fix the inner automorphisms U and V. Second problem concerns the possible values of the parameter t. The sum over all allowed values of t is expected to appear when one integrates over configuration space. One might hope is that it is possible to forget all the details of space-time surfaces and perform the integral explicitly to get the analog of propagator $1/(L_0 + i\epsilon)$ as in the case of string models.

To sum up, it is obvious that the option based on III_1 factors is very complex even in the case that propagator reduces to stringy propagator and the question how to extend S-matrix elements to other number fields looks formidable.

6.5 Feynman diagrams as higher level particles and their scattering as dynamics of self consciousness

The hierarchy of imbeddings of hyper-finite factors of II_1 as counterpart for many-sheeted space-time lead inevitably to the idea that this hierarchy corresponds to a hierarchy of generalized

Feynman diagrams for which Feynman diagrams at a given level become particles at the next level. Accepting this idea, one is led to ask what kind of quantum states these Feynman diagrams correspond, how one could describe interactions of these higher level particles, what is the interpretation for these higher level states, and whether they can be detected.

6.5.1 Jones inclusions as analogs of space-time surfaces

The idea about space-time as a 4-surface replicates itself at the level of operator algebra and state space in the sense that Jones inclusion can be seen as a representation of the operator algebra $\mathcal{N}$ as infinite-dimensional linear sub-space (surface) of the operator algebra $\mathcal{M}$. This encourages to think that generalized Feynman diagrams could correspond to image surfaces in II_1 factor having identification as kind of quantum space-time surfaces.

Suppose that the modular S-matrices are representable as the inner automorphisms $\Delta(\mathcal{M}_k^{it}$ assigned to the external lines of Feynman diagrams. This would mean that $\mathcal{N} \subset \mathcal{M}_k$ moves inside $calM_k$ along a geodesic line determined by the inner automorphism. At the vertex the factors $calM_k$ to fuse along $\mathcal{N}$ to form a Connes tensor product. Hence the copies of $\mathcal{N}$ move inside $\mathcal{M}_k$ like incoming 3-surfaces in H and fuse together at the vertex. Since all $\mathcal{M}_k$ are isomorphic to a universal factor $\mathcal{M}$, many-sheeted space-time would have a kind of quantum image inside II_1 factor consisting of pieces which are $d = \mathcal{M} : \mathcal{N}/2$-dimensional quantum spaces according to the identification of the quantum space as subspace of quantum group to be discussed later. In the case of partonic Clifford algebras the dimension would be indeed $d \leq 2$.

6.5.2 The hierarchy of Jones inclusions defines a hierarchy of S-matrices

It is possible to assign to a given Jones inclusion $\mathcal{N} \subset \mathcal{M}$ an entire hierarchy of Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2$..., $\mathcal{M}_0 = N$, $\mathcal{M}_1 = M$. A possible interpretation for these inclusions would be as a sequence of topological condensations.

This sequence also defines a hierarchy of Feynman diagrams inside Feynman diagrams. The factor $\mathcal{M}$ containing the Feynman diagram having as its lines the unitary orbits of $\mathcal{N}$ under $\Delta_{\mathcal{M}}$ becomes a parton in $\mathcal{M}_1$ and its unitary orbits under $\Delta_{\mathcal{M}_1}$ define lines of Feynman diagrams in M_1. The concrete representation for S-matrix or projection of it to some subspace as entanglement coefficients of partons at the ends of a braid assignable to the space-like 3-surface representing a vertex of a higher level Feynman diagram. In this manner quantum dynamics would be coded and simulated by quantum states.

The outcome can be said to be a hierarchy of Feynman diagrams within Feynman diagrams, a fractal structure for which many particle scattering events at a given level become particles at the next level. The particles at the next level represent dynamics at the lower level: they have the property of "being about" representing perhaps the most crucial element of conscious experience. Since net conserved quantum numbers can vanish for a system in TGD Universe, this kind of hierarchy indeed allows a realization as zero energy states. Crossing symmetry can be understood in terms of this picture and has been applied to construct a model for S-matrix at high energy limit [C4].

One might perhaps say that quantum space-time corresponds to a double inclusion and that further inclusions bring in N-parameter families of space-time surfaces.

6.5.3 Higher level Feynman diagrams

The lines of Feynman diagram in $\mathcal{M}_{n+1}$ are geodesic lines representing orbits of $\mathcal{M}_n$ and this kind of lines meet at vertex and scatter. The evolution along lines is determined by $\Delta_{\mathcal{M}_{n+1}}$. These lines contain within themselves $\mathcal{M}_n$ Feynman diagrams with similar structure and the hierarchy continues down to the lowest level at which ordinary elementary particles are encountered.

For instance, the generalized Feynman diagrams at the second level are ribbon diagrams obtained by thickening the ordinary diagrams in the new time direction. The interpretation as ribbon diagrams crucial for topological quantum computation and suggested to be realizable in terms of zero energy states in [E9] is natural. At each level a new time parameter is introduced so that the dimension of the diagram can be arbitrarily high. The dynamics is not that of ordinary surfaces but the dynamics induced by the $\Delta_{\mathcal{M}_n}$.

6.5.4 Quantum states defined by higher level Feynman diagrams

The intuitive picture is that higher level quantum states corresponds to the self reflective aspect of existence and must provide representations for the quantum dynamics of lower levels in their own structure. This dynamics is characterized by S-matrix whose elements have representation in terms of Feynman diagrams.

1. These states correspond to zero energy states in which initial states have "positive energies" and final states have "negative energies". The net conserved quantum numbers of initial and final state partons compensate each other. Gravitational energies, and more generally gravitational quantum numbers defined as absolute values of the net quantum numbers of initial and final states do not vanish. One can say that thoughts have gravitational mass but no inertial mass.

2. States in sub-spaces of positive and negative energy states are entangled with entanglement coefficients given by S-matrix at the level below.

To make this more concrete, consider first the simplest non-trivial case. In this case the particles can be characterized as ordinary Feynman diagrams, or more precisely as scattering events so that the state is characterized by $\hat{S} = P_{in}SP_{out}$, where S is S-matrix and P_{in} *resp.* P_{out} is the projection to a subspace of initial *resp.* final states. An entangled state with the projection of S-matrix giving the entanglement coefficients is in question.

The larger the domains of projectors P_{in} and P_{out}, the higher the representative capacity of the state. The norm of the non-normalized state $\hat{S}$ is $Tr(\hat{S}\hat{S}^\dagger) \leq 1$ for II_1 factors, and at the limit $\hat{S} = S$ the norm equals to 1. Hence, by II_1 property, the state always entangles infinite number of states, and can in principle code the entire S-matrix to entanglement coefficients.

The states in which positive and negative energy states are entangled by a projection of S-matrix might define only a particular instance of states for which conserved quantum numbers vanish. The model for the interaction of Feynman diagrams discussed below applies also to these more general states.

6.5.5 The interaction of $\mathcal{M}_n$ Feynman diagrams at the second level of hierarchy

What constraints can one pose to the higher level reactions? How Feynman diagrams interact? Consider first the scattering at the second level of hierarchy ($\mathcal{M}_1$), the first level $\mathcal{M}_0$ being assigned to the interactions of the ordinary matter.

1. Conservation laws pose constraints on the scattering at level $\mathcal{M}_1$. The Feynman diagrams can transform to new Feynman diagrams only in such a manner that the net quantum numbers are conserved separately for the initial positive energy states and final negative energy states of the diagram. The simplest assumption is that positive energy matter and negative energy matter know nothing about each other and effectively live in separate worlds. The scattering matrix form Feynman diagram like states would thus be simply the tensor product $S \otimes S^\dagger$, where S is the S-matrix characterizing the lowest level interactions. Reductionism would be realized in the sense that, apart from the new elements brought in by $\Delta_{\mathcal{M}_n}$ defining single

particle free dynamics, the lowest level would determine in principle everything occurring at the higher level providing representations about representations about... for what occurs at the basic level. The lowest level would represent the physical world and higher levels the theory about it.

2. The description of hadronic reactions in terms of partons serves as a guide line when one tries to understand higher level Feynman diagrams. The fusion of hadronic space-time sheets corresponds to the vertices $\mathcal{M}_1$. In the vertex the analog of parton plasma is formed by a process known as parton fragmentation. This means that the partonic Feynman diagrams belonging to disjoint copies of $\mathcal{M}_0$ find themselves inside the same copy of $\mathcal{M}_0$. The standard description would apply to the scattering of the initial *resp.* final state partons.

3. After the scattering of partons hadronization takes place. The analog of hadronization in the recent case is the organization of the initial and final state partons to groups I_i and F_i such that the net conserved quantum numbers are same for I_i and F_i. These conditions can be satisfied if the interactions in the plasma phase occur only between particles belonging to the clusters labelled by the index i. Otherwise only single particle states in $\mathcal{M}_1$ would be produced in the reactions in the generic case. The cluster decomposition of S-matrix to a direct sum of terms corresponding to partitions of the initial state particles to clusters which do not interact with each other obviously corresponds to the "hadronization". Therefore no new dynamics need to be introduced.

4. One cannot avoid the question whether the parton picture about hadrons indeed corresponds to a higher level physics of this kind. This would require that hadronic space-time sheets carry the net quantum numbers of hadrons. The net quantum numbers associated with the initial state partons would be naturally identical with the net quantum numbers of hadron. Partons and they negative energy conjugates would provide in this picture a representation of hadron about hadron. This kind of interpretation of partons would make understandable why they cannot be observed directly. A possible objection is that the net gravitational mass of hadron would be three times the gravitational mass deduced from the inertial mass of hadron if partons feed their gravitational fluxes to the space-time sheet carrying Earth's gravitational field.

5. This picture could also relate to the suggested duality between string and parton pictures [E2]. In parton picture hadron is formed from partons represented by space-like 2-surfaces X_i^2 connected by join along boundaries bonds. In string picture partonic 2-surfaces are replaced with string orbits. If one puts positive and negative energy particles at the ends of string diagram one indeed obtains a higher level representation of hadron. If these pictures are dual then also in parton picture positive and negative energies should compensate each other. Interestingly, light-like 3-D causal determinants identified as orbits of partons could be interpreted as orbits of light like string word sheets with "time" coordinate varying in space-like direction.

6.5.6 Scattering of Feynman diagrams at the higher levels of hierarchy

This picture generalizes to the description of higher level Feynman diagrams.

1. Assume that higher level vertices have recursive structure allowing to reduce the Feynman diagrams to ordinary Feynman diagrams by a procedure consisting of finite steps.

2. The lines of diagrams are classified as incoming or outgoing lines according to whether the time orientation of the line is positive or negative. The time orientation is associated with the time parameter t_n characterizing the automorphism $\Delta_{\mathcal{M}_\backslash}^{it_n}$. The incoming and outgoing

net quantum numbers compensate each other. These quantum numbers are basically the quantum numbers of the state at the lowest level of the hierarchy.

3. In the vertices the $\mathcal{M}_{n+1}$ particles fuse and $\mathcal{M}_n$ particles form the analog of quark gluon plasma. The initial and final state particles of $\mathcal{M}_n$ Feynman diagram scatter independently and the S-matrix S_{n+1} describing the process is tensor product $S_n \otimes S_n^\dagger$. By the clustering property of S-matrix, this scattering occurs only for groups formed by partons formed by the incoming and outgoing particles $\mathcal{M}_n$ particles and each outgoing $\mathcal{M}_{n+1}$ line contains and irreducible $\mathcal{M}_n$ diagram. By continuing the recursion one finally ends down with ordinary Feynman diagrams.

6.5.7 A connection with TGD inspired theory of consciousness

The implications of this picture TGD inspired theory of consciousness are rather breathtaking.

1. The hierarchy of self representations and the reduction of their quantum dynamics to the dynamics of the material world apart from the effects brought in by the automorphisms Δ_{M_n} determining the free propagation of thoughts, would mean a concrete calculable theory for the quantum dynamics of cognition. My sincere hope is however that no one would ever christen these states "particles of self consciousness". These states are not conscious, consciousness would be in the quantum jump between these states.

2. Cognitive representations would possess "gravitational" charges, in particular gravitational mass, so that thoughts could be put into "gravitational scale". I have proposed that "gravitational" charges correspond to classical charges characterizing the systems at space-time level as opposed to quantum charges.

3. As found, even hadrons could form self representations usually assigned with human brain. This is certainly something that neuroscientist would not propose but conforms with the basic prediction of TGD inspired theory of consciousness [10] about infinite self hierarchy involving cognitive representations at all levels of the hierarchy [K1].

4. The TGD inspired model of topological quantum computation [E9] in terms of zero energy cognitive states inspired the proposal that the appearance of a representation and its negative energy conjugate could relate very intimately to the fact that DNA appears as double helices of a strand and its conjugate. This could also relate to the fact that binary structures are common in living matter.

5. One is forced to consider a stronger characterization of dark matter [D6, J6] as a matter at higher levels of the hierarchy with vanishing net inertial quantum numbers but with non-vanishing "gravitational" quantum numbers. We would detect dark matter via its "gravitational" charges. We would also experience it directly since our thoughts would be dark matter! The cosmological estimates for the proportion of dark matter and dark energy would give also estimate for the gravitational mass of thoughts in the Universe! This speculation is probably not quite correct since also more general entanglement than that defined by two-sided projections of S-matrix is possible between positive and negative energy states.

6.5.8 Cognitive entanglement as Connes tensor product

In the proposed construction the lowest level $\mathcal{N}$ represents matter and higher levels give cognitive representations. The ordinary tensor product $S \otimes S$ and its tensor powers define a hierarchy of S-matrices and the two-sided projections of these S-matrices in turn define entanglement coefficients for positive and negative energy states at various levels of hierarchy.

The following arguments show that the cognitive tensor product restricted to projections of S-matrix corresponds to the so called Connes tensor product appearing naturally in the hierarchy of Jones inclusions.A slight generalization of earlier scenario predicting matter-mind type transitions is forced by this identification and a beautiful interpretation for these transitions in terms of space-time correlates emerges.

1. Connes tensor product

Connes [38, 45] has introduced a variant of tensor product allowing to express the union $\cup M_i$, where M_i the inclusion hierarchy as infinite tensor product $M \otimes_N M \otimes_N \otimes ...$ The Connes tensor product $\otimes_N$ differs from the standard tensor product and is obtained by requiring that in the Connes tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ the condition $n\xi_1 \otimes_{\mathcal{N}} \xi_2 = \xi_1 \otimes_{\mathcal{N}} n\xi_2$ for all $n \in \mathcal{N}$ holds true. Connes tensor product means forces to replace ordinary statistics with braid statistics. The physical interpretation proposed by Connes is that this tensor product could make sense when N represents observables common to the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Later it will be found that TGD suggests quite different interpretation.

Connes tensor product makes sense also for finite-dimensional right and left modules. Consider the spaces $M_{n\times q}$ of $n\times q$-matrices and $M_{p\times n}$ of $p\times n$ matrices for which $n\times n$ matrix algebra $M_{n\times n}$ acts as a left *resp.* right multiplier. The tensor product $\otimes_N$ for these matrices is the ordinary matrix product of $m_{p\times n} \times m_{n\times q}$ and belongs to $M_{p\times q}$ so that the dimension of tensor product space is much lower than $m \times q \times n^2$ and does not depend on n. For Jones inclusion N takes the role of $M_{n\times n}$ and since M can be regarded as β-dimensional $\mathcal{N}$-module, tensor product can be said to give $\sqrt{\beta} \times \sqrt{\beta}$-dimensional matrices with $\mathcal{N}$ valued entries. In particular, the inclusion sequence is an infinite tensor product of $\sqrt{\beta} \times \sqrt{\beta}$-dimensional matrices.

2. Does Connes tensor product generate cognitive entanglement?

One can wonder why the entanglement coefficients between positive and negative energy states should be restricted to the projections of S-matrix. The obvious guess is that it gives rise to an entanglement equivalent with Connes tensor product so that the action of N on initial state is equivalent with its action on the final state. This indeed seems to be the case. The basic symmetry of Connes tensor product translates to the possibility to move an operator creating particles in initial state to final state by conjugating it: this is nothing but crossing symmetry characterizing S-matrix. Thus Connes tensor product generates zero energy states providing a hierarchy of cognitive representations.

3. Do transitions between different levels of cognitive hierarchy occur?

The following arguments suggest that the proposed hierarchy of cognitive representations is not exhaustive.

1. Only tensor powers of S involving $(2^n)^{th}$ powers of S appear in the cognitive hierarchy as it is constructed. Connes tensor product representation of $\cup_i M_i$ would however suggest that all powers of S appear.

2. There is no reason to restrict the states to positive energy states in TGD Universe. In fact, the states of the entire Universe have zero energy. Thus much more general zero energy states are possible in TGD framework than those for which entanglement is given by a projection of S-matrix, and they occur already at the lowest level of the hierarchy.

On basis of these observations there is no reason to exclude transitions between different levels of the cognitive hierarchy transforming ordinary tensor product of positive and negative energy states with vanishing conserved quantum numbers to a Connes tensor product involving only the projection of S-matrix as entanglement coefficients. These transitions would give rise to S-matrices connecting different levels and thus fill the gaps in the spectrum of allowed tensor powers of S.

4. Space-time correlates for the matter-to-mind transitions

Allowing somewhat poetic language, the scatterings in question would represent kind of matter-to-mind transitions, enlightment, or transition to a Buddha state. At space-time level zero energy matter would correspond to positive and negative energy states with a space-like separation whereas cognitive states would correspond to positive and negative energy states with a time-like separation. By the failure of the complete classical determinism time like entanglement makes sense but due to the fact determinism is not completely lost, entanglement could be of a very special kind only, and S-matrix could appear as entanglement coefficients.

Light-like causal determinants (CDs) identifiable as orbits of both space-like partonic 2-surfaces and light-like stringy surfaces, can be said to represent both matter and mind. According to the proposal of [E9], light-like CDs would correspond to both programs and computers for topological quantum computation, and matter-mind transformation would be also involved with the realization of the genetic code both as cognitive and material structures. This would support the view that the stringy 2-surfaces in the foliation of the space-time surface are time-like in the interior of the space-time sheet (or more generally, outside light-like causal determinants) and light-like causal determinants correspond to critical line between matter and mind.

7 Jones inclusions and cognitive consciousness

Configuration space spinors have a natural interpretation in terms of a quantum version of Boolean algebra. Beliefs of various kinds are the basic element of cognition and obviously involve a representation of the external world or part of it as states of the system defining the believer. Jones inclusions mediating unitary mappings between the spaces of configuration spaces spinors of two systems are excellent candidates for these maps, and it is interesting to find what one kind of model for beliefs this picture leads to.

The resulting quantum model for beliefs provides a cognitive interpretation for quantum groups and predicts a universal spectrum for the probabilities that a given belief is true. This spectrum depends only on the integer n characterizing the quantum phase $q = exp(i2\pi/n)$ characterizing the Jones inclusion. For $n \neq \infty$ the logic is inherently fuzzy so that absolute knowledge is impossible. $q = 1$ gives ordinary quantum logic with qbits having precise truth values after state function reduction.

7.1 Logic, beliefs, and spinor fields in the world of classical worlds

Beliefs can be characterized as Boolean value maps $\beta_i(p)$ telling whether i believes in proposition p or not. Additional structure is brought in by introducing the map $\lambda_i(p)$ telling whether p is true or not in the environment of i. The task is to find quantum counterpart for this model.

7.1.1 Configuration space spinors as logic statements

In TGD framework the infinite-dimensional configuration space (CH) spinor fields defined in CH, the "world of classical worlds", describe quantum states of the Universe [B4]. CH spinor field can be regarded as a state in infinite-dimensional Fock space and are labelled by a collection of various two valued indices like spin and weak isospin. The interpretation is as a collection of truth values of logic statements one for each fermionic oscillator operator in the state. For instance, spin up and down would correspond to two possible truth values of a proposition characterized by other quantum numbers of the mode.

The hierarchy of space-time sheet could define a physical correlate for the hierarchy of higher order logics (statements about statements about...). The space-time sheet containing N fermions topologically condensed at a larger space-time sheet behaves as a fermion or boson depending on whether N is odd or even. This hierarchy has also a number theoretic counterpart: the construction of infinite primes [E3] corresponds to a repeated second quantization of a super-symmetric quantum field theory.

7.1.2 Quantal description of beliefs

The question is whether TGD inspired theory of consciousness allows a fundamental description of beliefs.

1. Beliefs define a model about some subsystem of universe constructed by the believer. This model can be understood as some kind of representation of real word in the state space representing the beliefs.

2. One can wonder what is the difference between real and p-adic variants of CH spinor fields and whether they could represent reality and beliefs about reality. CH spinors (as opposed to spinor fields) are constructible in terms of fermionic oscillator operators and seem to be universal in the sense that one cannot speak about p-adic and real CH spinors as different objects. Real/ p-adic spinor fields however have real/p-adic space-time sheets as arguments. This would suggest that there is no fundamental difference between the logic statements represented by p-adic and real CH spinors.

These observations suggest a more concrete view about how beliefs emerge physically.

The idea that p-adic CH spinor fields could serve as representations of beliefs and real CH spinor fields as representations of reality looks very nice but the fact that the outcomes of p-adic-to-real phase transition and its reversal are highly non-predictable does not support it as such.

Quantum statistical determinism could however come into rescue. Belief could be represented as an ensemble of p-adic mental images resulting in transitions of real mental images representing reality to p-adic states. p-Adic ensemble average would represent the belief.

It is not at all clear whether real-to-padic transitions can occur at high enough rate since p-adic-to-real transition are expected to be highly irreversible. The real initial states much have nearly vanishing quantum numbers emitted in the transition to p-adic state to guarantee conservation laws (p-adic conservation laws hold true only piecewise since conserved quantities are pseudo constants). The system defined by an ensemble of real Boolean mental images representing reality would automatically generate a p-adic variant representing a belief about reality.

p-Adic CH spinors can also represent the cognitive aspects of intention whereas p-adic space-time sheets would represent its geometric aspects reflected in sensory experience.p-Adic space-time sheet could also serve only as a space-time correlate for the fundamental representation of intention in terms of p-adic CH spinor field. This view is consistent with the proposed identification of beliefs since the transitions associated with intentions *resp.* beliefs would be p-adic-to-real *resp.* real-to-padic.

7.2 Jones inclusions for hyperfinite factors of type II_1 as a model for symbolic and cognitive representations

Consider next a more detailed model for how cognitive representations and beliefs are realized at quantum level. This model generalizes trivially to symbolic representations.

The Clifford algebra of gamma matrices associated with CH spinor fields corresponds to a von Neumann algebra known as hyper-finite factor of type II_1. The mathematics of these algebras is extremely beautiful and reproduces basic mathematical structures of modern physics (conformal field theories, quantum groups, knot and braid groups,....) from the mere assumption that the world of classical worlds possesses infinite-dimensional Kähler geometry and allows spinor structure.

The almost defining feature is that the infinite-dimensional unit matrix of the Clifford algebra in question has by definition unit trace. Type II_1 factors allow also what are known as Jones inclusions of Clifford algebras $\mathcal{N} \subset \mathcal{M}$. What is special to II_1 factors is that the induced unitary mappings between spinor spaces are genuine inclusions rather than 1-1 maps.

The S-matrix associated with the real-to-p-adic quantum transition inducing belief from reality would naturally define Jones inclusion of CH Clifford algebra $\mathcal{N}$ associated with the real space-time sheet to the Clifford algebra $\mathcal{M}$ associated with the p-adic space-time sheet. The moduli squared of S-matrix elements would define probabilities for pairs or real and belief states.

In Jones inclusion $\mathcal{N} \subset \mathcal{M}$ the factor $\mathcal{N}$ is included in factor $\mathcal{M}$ such that $\mathcal{M}$ can be expressed as $\mathcal{N}$-module over quantum space $\mathcal{M}/\mathcal{N}$ which has fractal dimension given by Jones index $\mathcal{M} : \mathcal{N} = 4cos^2(\pi/n) \leq 4$, $n = 3, 4, ...$ varying in the range $[1, 4]$. The interpretation is as the fractal dimension corresponding to a dimension of Clifford algebra acting in $d = \sqrt{\mathcal{M} : \mathcal{N}}$-dimensional spinor space: d varies in the range $[1, 2]$. The interpretation in terms of a quantal variant of logic is natural.

7.2.1 Probabilistic beliefs

For $\mathcal{M} : \mathcal{N} = 4$ ($n = \infty$) the dimension of spinor space is $d = 2$ and one can speak about ordinary 2-component spinors with $\mathcal{N}$-valued coefficients representing generalizations of qubits. Hence the inclusion of a given $\mathcal{N}$-spinor as M-spinor can be regarded as a belief on the proposition and for the decomposition to a spinor in N-module $\mathcal{M}/\mathcal{N}$ involves for each index a choice $\mathcal{M}/\mathcal{N}$ spinor component selecting super-position of up and down spins. Hence one has a superposition of truth values in general and one can speak only about probabilistic beliefs. It is not clear whether one can choose the basis in such a manner that $\mathcal{M}/\mathcal{N}$ spinor corresponds always to truth value 1. Since CH spinor field is in question and even if this choice might be possible for a single 3-surface, it need not be possible for deformations of it so that at quantum level one can only speak about probabilistic beliefs.

7.2.2 Fractal probabilistic beliefs

For $d < 2$ the spinor space associated with $\mathcal{M}/\mathcal{N}$ can be regarded as quantum plane having complex quantum dimension d with two non-commuting complex coordinates z^1 and z^2 satisfying $z^1z^2 = qz^2z^1$ and $\overline{z^1z^2} = \overline{q}\overline{z^2z^1}$. These relations are consistent with hermiticity of the real and imaginary parts of z^1 and z^2 which define ordinary quantum planes [C7]. Hermiticity also implies that one can identify the complex conjugates of z^i as Hermitian conjugates.

The further commutation relations $[z^1, \overline{z^2}] = [z^2, \overline{z^1}] = 0$ and $[z^1, \overline{z^1}] = [z^2, \overline{z^2}] = r$ give a closed algebra satisfying Jacobi identities. One could argue that $r \geq 0$ should be a function $r(n)$ of the quantum phase $q = exp(i2\pi/n)$ vanishing at the limit $n \to \infty$ to guarantee that the algebra becomes commutative at this limit and truth values can be chosen to be non-fuzzy. $r = sin(\pi/n)$ would be the simplest choice. As will be found, the choice of $r(n)$ does not however affect at all the spectrum for the probabilities of the truth values. $n = \infty$ case corresponding to non-fuzzy quantum logic is also possible and must be treated separately: it corresponds to Kac Moody algebra instead of quantum groups.

The non-commutativity of complex spinor components means that z^1 and z^2 are not independent coordinates: this explains the reduction of the number of the effective number of truth values to $d < 2$. The maximal reduction occurs to $d = 1$ for $n = 3$ so that there is effectively only single truth value and one could perhaps speak about taboo or dogma or complete disappearance of the notions of truth and false (this brings in mind reports about meditative states: in fact $n = 3$ corresponds to a phase in which Planck constant becomes infinite so that the system is maximally quantal).

As non-commuting operators the components of d-spinor are not simultaneously measurable for $d < 2$. It is however possible to measure simultaneously the operators describing the probabilities $z^1\overline{z^1}$ and $z^2\overline{z^2}$ for truth values since these operators commute. An inherently fuzzy Boolean logic would be in question with the additional feature that the spinorial counterparts of statement and its negation cannot be regarded as independent observables although the corresponding probabilities satisfy the defining conditions for commuting observables.

If one can speak of a measurement of probabilities for $d < 2$, it differs from the ordinary quantum measurement in the sense that it cannot involve a state function reduction to a pure qubit meaning irreducible quantal fuzziness. One could speak of fuzzy qbits or fqbits (or quantum qbits) instead of qbits. This picture would provide the long sought interpretation for quantum groups.

The previous picture applies to all representations $M_1 \subset M_2$, where M_1 and M_2 denote either real or p-adic Clifford algebras for some prime p. For instance, real-real Jones inclusion could be interpreted as symbolic representations assignable to a unitary mapping of the states of a subsystem M_1 of the external world to the state space M_2 of another real subsystem. $p_1 \rightarrow p_2$ unitary inclusions would in turn map cognitive representations to cognitive representations. There is a strong temptation to assume that these Jones inclusions define unitary maps realizing universe as a universal quantum computer mimicking itself at all levels utilizing cognitive and symbolic representations. Subsystem-system inclusion would naturally define one example of Jones inclusion.

7.2.3 The spectrum of probabilities of truth values is universal

It is actually possible to calculate the spectrum of the probabilities of truth values with rather mild additional assumptions.

1. Since the Hermitian operators $X_1 = (z^1\overline{z^1} + \overline{z^1}z^1)/2$ and $X_2 = (z^2\overline{z^2} + \overline{z^2}z^2)/2$ commute, physical states can be chosen to be eigen states of these operators and it is possible to assign to the truth values probabilities given by $p_1 = X_1/R^2$ and $p_2 = X_2/R^2$, $R^2 = X_1 + X_2$.

2. By introducing the analog of the harmonic oscillator vacuum as a state $|0\rangle$ satisfying $z^1|0\rangle = z^2|0\rangle = 0$, one obtains eigen states of X_1 and X_2 as states $|n_1, n_2\rangle = \overline{z^1}^{n_1}\overline{z^2}^{n_2}|0\rangle$, $n_1 \geq 0, n_2 \geq 0$. The eigenvalues of X_1 and X_2 are given by a modified harmonic oscillator spectrum as $(1/2 + n_1 q^{n_2})r$ and $(1/2 + n_2 q^{n_1})r$. The reality of eigenvalues (hermiticity) is guaranteed if one has $n_1 = N_1 n$ and $n_1 = N_2 n$ and implies that the spectrum of eigen states gets increasingly thinner for $n \rightarrow \infty$. This must somehow reflect the fractal dimension. The fact that large values of oscillator quantum numbers n_1 and n_2 correspond to the classical limit suggests that modulo condition guarantees approximate classicality of the logic for $n \rightarrow \infty$.

3. The probabilities p_1 and p_2 for the truth values given by $(p_1, p_2) = (1/2+N_1 n, 1/2+N_2 n)/[1+(N_1 + N_2)n]$ are rational and allow an interpretation as both real and p-adic numbers. All states are are inherently fuzzy and only at the limits $N_1 \gg N_2$ and $N_2 \gg N_1$ non-fuzzy states result. As noticed, $n = \infty$ must be treated separately and corresponds to an ordinary non-fuzzy qbit logic. At $n \rightarrow \infty$ limit one has $(p_1, p_2) = (N_1, N_2)/(N_1, N_2)$: at this limit $N_1 = 0$ or $N_2 = 0$ states are non-fuzzy.

7.2.4 How to define variants of belief quantum mechanically?

Probabilities of true and false for Jones inclusion characterize the plausibility of the belief and one can ask whether this description is enough to characterize states such as knowledge, misbelief, doubt, delusion, and ignorance. The truth value of $\beta_i(p)$ is determined by the measurement of probability assignable to Jones inclusion on the p-adic side. The truth value of $\lambda_i(p)$ is determined by a similar measurement on the real side. β and λ appear completely symmetrically and one can consider all kinds of triplets $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ assuming that there exist unitary S-matrix like maps mediating a sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ of Jones inclusions. Interestingly, the hierarchies of Jones inclusions are a key concept in the theory of hyper-finite factors of type II_1 and pair of inclusions plays a fundamental role.

Let us restrict the consideration to the situation when $\mathcal{M}_1$ corresponds to a real subsystem of the external world, $\mathcal{M}_2$ its real representation by a real subsystem, and $\mathcal{M}_3$ to p-adic cognitive representation of M_3. Assume that both real and p-adic sides involve a preferred state basis for qubits representing truth and false.

Assume first that both $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\mathcal{M}_2 \subset \mathcal{M}_3$ correspond to $d = 2$ case for which ordinary quantum measurement or truth value is possible giving outcome true or false. Assume further that the truth values have been measured in both M_2 and M_3.

1. Knowledge corresponds to the proposition $\beta_i(p) \wedge \lambda_i(p)$.

2. Misbelief to the proposition $\beta_i(p) \wedge \neq \lambda_i(p)$.
   Knowledge and misbelief would involve both the measurement of real and p-adic probabilities
   .

3. Assume next that one has $d < 2$ form $\mathcal{M}_2 \subset \mathcal{M}_3$. Doubt can be regarded neither belief or disbelief: $\beta_i(p) \wedge \neq \beta_i(\neq p)$: belief is inherently fuzzy although proposition can be non-fuzzy.

   Assume next that truth values in $\mathcal{M}_1 \subset \mathcal{M}_2$ inclusion corresponds to $d < 2$ so that the basic propositions are inherently fuzzy.

4. Delusion is a belief which cannot be justified: $\beta_i(p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. This case is possible if $d = 2$ holds true for $\mathcal{M}_2 \subset \mathcal{M}_3$. Note that also misbelief that cannot be shown wrong is possible.
   In this case truth values cannot be quantum measured for $\mathcal{M}_1 \subset \mathcal{M}_2$ but can be measured for $\mathcal{M}_2 \subset \mathcal{M}_3$. Hence the states are products of pure $\mathcal{M}_3$ states with fuzzy $\mathcal{M}_2$ states.

5. Ignorance corresponds to the proposition $\beta_i(p) \wedge \neq \beta_i(\neq p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. Both real representational states and belief states are inherently fuzzy.

Quite generally, only for $d_1 = d_2 = 2$ ideal knowledge and ideal misbelief are possible. Fuzzy beliefs and logics approach to ordinary one at the limit $n \rightarrow \infty$, which according to the proposal of [D6] corresponds to the ordinary value of Planck constant. For other cases these notions are only approximate and quantal approach allows to characterize the goodness of the approximation. A new kind of inherent quantum uncertainty of knowledge is in question and one could speak about a Uncertainty Principle for cognition and symbolic representations. Also the unification of symbolic and various kinds of cognitive representations deserves to be mentioned.

7.3 Intentional comparison of beliefs by topological quantum computation?

Intentional comparison would mean that for a given initial state also the final state of the quantum jump is fixed. This requires the ability to engineer S-matrix so that it leads from a given state to single state only. Any S-matrix representing permutation of the initial states fulfills these conditions. This condition is perhaps unnecessarily strong.

Quantum computation is basically the engineering of S-matrix so that it represents a superposition of parallel computations. In TGD framework topological quantum computation based on the braiding of magnetic flux tubes would be represented as an evolution characterized by braid [E9]. The dynamical evolution would be associated with light-like boundaries of braids. This evolution has dual interpretations either as a limit of time evolution of quantum state (program running) or a quantum state satisfying conformal invariance constraints (program code).

The dual interpretation would mean that conformally invariant states are equivalent with engineered time evolutions and topological computation realized as braiding connecting the quantum states to be compared (beliefs represented as many-fermion states at the boundaries of magnetic flux tubes) could give rise to conscious computational comparison of beliefs. The complexity of braiding would give a measure for how much the states to be compared differ.

Note that quantum computation is defined by a unitary map which could also be interpreted as symbolic representation of states of system M_1 as states of system M_2 mediated by the braid of join along boundaries bonds connecting the two space-time sheets in question and having light-like boundaries. These considerations suggest that the idea about S-matrix of the Universe should be generalized so that the dynamics of the Universe is dynamics of mimicry described by an infinite collection of fermionic S-matrices representable in terms of Jones inclusions.

7.4 The stability of fuzzy qbits and quantum computation

The stability of fqbits against state function reduction might have deep implications for quantum computation since quantum spinors would be stable against state function reduction induced by the perturbations inducing de-coherence in the normal situation. If this is really true, and if the only dangerous perturbations are those inducing the phase transition to qbits, the implications for quantum computation could be dramatic. Of course, the rigidity of qbits could be just another way to say that topological quantum computations are stable against thermal perturbations not destroying anyons [E9].

The stability of fqbits could also be another manner to state the stability of rational, or more generally algebraic, bound state entanglement against state function reduction, which is one of the basic hypothesis of TGD inspired theory of consciousness [H1]. For sequences of Jones inclusions or equivalently, for multiple Connes tensor products, one would obtain tensor products of quantum spinors making possible arbitrary complex configurations of fqbits. Anyonic braids in topological quantum computation would have interpretation as representations for this kind of tensor products.

7.5 Fuzzy quantum logic and possible anomalies in the experimental data for the EPR-Bohm experiment

The experimental data for EPR-Bohm experiment [62] excluding hidden variable interpretations of quantum theory. What is less known that the experimental data indicates about possibility of an anomaly challenging quantum mechanics [80]. The obvious question is whether this anomaly might provide a test for the notion of fuzzy quantum logic inspired by the TGD based quantum measurement theory with finite measurement resolution.

7.5.1 The anomaly

The experimental situation involves emission of two photons from spin zero system so that photons have opposite spins. What is measured are polarizations of the two photons with respect to polarization axes which differ from standard choice of this axis by rotations around the axis of photon momentum characterized by angles α and β. The probabilities for observing polarizations (i,j), where i,j is taken Z_2 valued variable for a convenience of notation are $P_{ij}(\alpha,\beta)$, are predicted to be $P_{00} = P_{11} = cos^2(\alpha-\beta)/2$ and $P_{01} = P_{10} = sin^2(\alpha-\beta)/2$.

Consider now the discrepancies.

1. One has four identities $P_{i,i} + P_{i,i+1} = P_{ii} + P_{i+1,i} = 1/2$ having interpretation in terms of probability conservation. Experimental data of [62] are not consistent with this prediction

[81] and this is identified as the anomaly.

2. The QM prediction $E(\alpha,\beta) = \sum_i (P_{i,i} - P_{i,i+1}) = cos(2(\alpha-\beta)$ is not satisfied neither: the maxima for the magnitude of E are scaled down by a factor $\simeq .9$. This deviation is not discussed in [81].

Both these findings raise the possibility that QM might not be consistent with the data. It turns out that fuzzy quantum logic predicted by TGD and implying that the predictions for the probabilities and correlation must be replaced by ensemble averages, can explain anomaly b) but not anomaly a). A "mundane" explanation for anomaly a) is proposed.

7.5.2 Predictions of fuzzy quantum logic for the probabilities and correlations

1. The description of fuzzy quantum logic in terms statistical ensemble

The fuzzy quantum logic implies that the predictions $P_{i,j}$ for the probabilities should be replaced with ensemble averages over the ensembles defined by fuzzy quantum logic. In practice this means that following replacements should be carried out:

$$\begin{aligned} P_{i,j} &\rightarrow P^2 P_{i,j} + (1-P)^2 P_{i+1,j+1} \\ &+ P(1-P)\left[P_{i,j+1} + P_{i+1,j}\right] \ . \end{aligned} \tag{12}$$

Here P is one of the state dependent universal probabilities/fuzzy truth values for some value of n characterizing the measurement situation. The concrete predictions would be following

$$\begin{aligned} P_{0,0} = P_{1,1} &\rightarrow A\frac{cos^2(\alpha-\beta)}{2} + B\frac{sin^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{cos^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ P_{0,1} = P_{1,0} &\rightarrow A\frac{sin^2(\alpha-\beta)}{2} + B\frac{cos^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{sin^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ A &= P^2 + (1-P)^2 \ , \ B = 2P(1-P) \ . \end{aligned} \tag{13}$$

The prediction is that the graphs of probabilities as a function as function of the angle $\alpha - \beta$ are scaled by a factor $1 - 4P(1-P)$ and shifted upwards by $P(1-P)$. The value of P, and one might hope even the value of n labelling Jones inclusion and the integer m labelling the quantum state might be deducible from the experimental data as the upward shift. The basic prediction is that the maxima of curves measuring probabilities $P_(i,j)$ have minimum at $B/2 = P(1-P)$ and maximum is scaled down to $(A-B)/2 = 1/2 - 2P(1-P)$.

If the P is same for all pairs i,j, the correlation $E = \sum_i (P_{ii} - P_{i,i+1})$ transforms as

$$E(\alpha,\beta) \rightarrow \left[1 - 4P(1-P)\right] E(\alpha,\beta) \ . \tag{14}$$

Only the normalization of $E(\alpha,\beta)$ as a function of $\alpha - \beta$ reducing the magnitude of E occurs. In particular the maximum/minimum of E are scaled down from $E = \pm 1$ to $E = \pm(1 - 4P(1-P))$.

From the figure 1b) of [81] the scaling down indeed occurs for magnitudes of E with same amount for minimum and maximum. Writing $P = 1 - \epsilon$ one has $A - B \simeq 1 - 4\epsilon$ and $B \simeq 2\epsilon$ so that the maximum is in the first approximation predicted to be at $1-4\epsilon$. The graph would give $1-P\simeq\epsilon\simeq .025$. Thus the model explains the reduction of the magnitude for the maximum and minimum of E which was not however considered to be an anomaly in [80, 81].

A further prediction is that the identities $P(i,i)+P(i+1,i)=1/2$ should still hold true since one has $P_{i,i}+P_{i,i+1}=(A-B)/2+B=1$. This is implied also by probability conservation. The four curves corresponding to these identities do not however co-incide as the figure 6 of [81] demonstrates. This is regarded as the basic anomaly in [80, 81]. From the same figure it is also clear that below $\alpha-\beta<10$ degrees $P_{++}=P_{--}$ $\Delta P_{+-}=-\Delta P_{-+}$ holds true in a reasonable approximation. After that one has also non-vanishing ΔP_{ii} satisfying $\Delta P_{++}=-\Delta P_{--}$. This kind of splittings guarantee the identity $\sum_{ij}P_{ij}=1$. These splittings are not visible in E.

Since probability conservation requires $P_{ii}+P_{ii+1}=1$, a mundane explanation for the discrepancy could be that the failure of the conditions $P_{i,i}+P_{ii+1}=1$ means that the measurement efficiency is too low for P_{+-} and yields too low values of $P_{+-}+P_{--}$ and $P_{+-}+P_{++}$. The constraint $\sum_{ij}P_{ij}=1$ would then yield too high value for P_{-+}. Similar reduction of measurement efficiency for P_{++} could explain the splitting for $\alpha-\beta>10$ degrees.

Clearly asymmetry with respect to exchange of photons or of detectors is in question.

1. The asymmetry of two photon state with respect to the exchange of photons could be considered as a source of asymmetry. This would mean that the photons are not maximally entangled. This could be seen as an alternative "mundane" explanation.

2. The assumption that the parameter P is different for the detectors does not change the situation as is easy to check.

3. One manner to achieve splittings which resemble observed splittings is to assume that the value of the probability parameter P depends on the *polarization pair*: $P=P(i,j)$ so that one has $(P(-,+),P(+,-))=(P+\Delta,P-\Delta)$ and $(P(-,-),P(+,+))=(P+\Delta_1,P-\Delta_1)$. $\Delta\simeq .025$ and $\Delta_1\simeq\Delta/2$ could produce the observed splittings qualitatively. One would however always have $P(i,i)+P(i,i+1)\geq 1/2$. Only if the procedure extracting the correlations uses the constraint $\sum_{i,j}P_{ij}=1$ effectively inducing a constant shift of P_{ij} downwards an asymmetry of observed kind can result. A further objection is that there are no special reason for the values of $P(i,j)$ to satisfy the constraints.

2. Is it possible to say anything about the value of n in the case of EPR-Bohm experiment?

To explain the reduction of the maximum magnitudes of the correlation E from 1 to $\sim .9$ in the experiment discussed above one should have $p_1\simeq .9$. It is interesting to look whether this allows to deduce any information about the valued of n. At the limit of large values of N_in one would have $(N_1-N_2)/(N_1+N_2)\simeq .4$ so that one cannot say anything about n in this case. $(N_1,N_2)=(3,1)$ satisfies the condition exactly. For $n=3$, the smallest possible value of n, this would give $p_1\simeq .88$ and for $n=4$ $p_1=.41$. With high enough precision it might be possible to select between $n=3$ and $n=4$ options if small values of N_i are accepted.

7.6 One element field, quantum measurement theory and its q-variant, and the Galois fields associated with infinite primes

John Baez talked in This Weeks Finds (Week 259) [71] about one-element field - a notion inspired by the $q=exp(i2\pi/n)\rightarrow 1$ limit for quantum groups. This limit suggests that the notion of one-element field for which 0=1 - a kind of mathematical phantom for which multiplication and sum should be identical operations - could make sense. Physicist might not be attracted by this kind of identification.

In the following I want to articulate some comments from the point of view of quantum measurement theory and its generalization to q-measurement theory which I proposed for some years ago and which is represented above.

I also consider and alternative interpretation in terms of Galois fields assignable to infinite primes which form an infinite hierarchy. These Galois fields have infinite number of elements but the map to the real world effectively reduces the number of elements to 2: 0 and 1 remain different.

7.6.1 $q \rightarrow 1$ limit as transition from quantum physics to effectively classical physics?

The $q \rightarrow 1$ limit of quantum groups at q-integers become ordinary integers and n-D vector spaces reduce to n-element sets. For quantum logic the reduction would mean that 2^N-D spinor space becomes 2^N-element set. N qubits are replaced with N bits. This brings in mind what happens in the transition from wave mechanism to classical mechanics. This might relate in interesting manner to quantum measurement theory.

Strictly speaking, $q \rightarrow 1$ limit corresponds to the limit $q = exp(i2\pi/n)$, $n \rightarrow \infty$ since only roots of unity are considered. This also correspond to Jones inclusions at the limit when the discrete group Z_n or or its extension-both subgroups of $SO(3)$- to contain reflection has infinite elements. Therefore this limit where field with one element appears might have concrete physical meaning. Does the system at this limit behave very classically?

In TGD framework this limit can correspond to either infinite or vanishing Planck constant depending on whether one consider orbifolds or coverings. For the vanishing Planck constant one should have classicality: at least naively! In perturbative gauge theory higher order corrections come as powers of $g^2/4\pi\hbar$ so that also these corrections vanish and one has same predictions as given by classical field theory.

7.6.2 Q-measurement theory and $q \rightarrow 1$ limit

Q-measurement theory differs from quantum measurement theory in that the coordinates of the state space, say spinor space, are non-commuting. Consider in the sequel q-spinors for simplicity.

Since the components of quantum spinor do not commute, one cannot perform state function reduction. One can however measure the modulus squared of both spinor components which indeed commute as operators and have interpretation as probabilities for spin up or down. They have a universal spectrum of eigen values. The interpretation would be in terms of fuzzy probabilities and finite measurement resolution but may be in different sense as in case of HFF:s. Probability would become the observable instead of spin for q not equal to 1.

At $q \rightarrow 1$ limit quantum measurement becomes possible in the standard sense of the word and one obtains spin down or up. This in turn means that the projective ray representing quantum states is replaced with one of n possible projective rays defining the points of n-element set. For HFF:s of type II_1 it would be N-rays which would become points, N the included algebra. One might also say that state function reduction is forced by this mapping to *single* object at $q \rightarrow 1$ limit.

On might say that the set of orthogonal coordinate axis replaces the state space in quantum measurement. We do this replacement of space with coordinate axis all the time when at blackboard. Quantum consciousness theorist inside me adds that this means a creation of symbolic representations and that the function of quantum classical correspondences is to build symbolic representations for quantum reality at space-time level.

$q \rightarrow 1$ limit should have space-time correlates by quantum classical correspondence. A TGD inspired geometro-topological interpretation for the projection postulate might be that quantum measurement at $q \rightarrow 1$ limit corresponds to a leakage of 3-surface to a dark sector of imbedding space with $q \rightarrow 1$ (Planck constant near to 0 or ∞ depending on whether one has $n \rightarrow \infty$ covering or division of M^4 or CP_2 by a subgroup of $SU(2)$ becoming infinite cyclic - very roughly!) and

Hilbert space is indeed effectively replaced with n rays. For $q \neq 1$ one would have only probabilities for different outcomes since things would be fuzzy.

In this picture classical physics and classical logic would be the physical counterpart for the shadow world of mathematics and would result only as an asymptotic notion.

7.6.3 Could 1-element fields actually correspond to Galois fields associated with infinite primes?

Finite field G_p corresponds to integers modulo p and product and sum are taken only modulo p. An alternative representation is in terms of phases $exp(ik2\pi/p)$, $k = 0, ..., p-1$ with sum and product performed in the exponent. The question is whether could one define these fields also for infinite primes [E3] by identifying the elements of this field as phases $exp(ik2\pi/\Pi)$ with k taken to be finite integer and Π an infinite prime (recall that they form infinite hierarchy). Formally this makes sense. 1-element field would be replaced with infinite hierarchy of Galois fields with infinite number of elements!

The probabilities defined by components of quantum spinor make sense only as real numbers and one can indeed map them to real numbers by interpreting q as an ordinary complex number. This would give same results as $q \rightarrow 1$ limit and one would have effectively 1-element field but actually a Galois field with infinite number of elements.

If one allows k to be also infinite integer but not larger than than Π in the real sense, the phases $exp(ik2\pi/\Pi)$ would be well defined as real numbers and could differ from 1. All real numbers in the range $[-1, 1]$ would be obtained as values of $cos(k2\pi/\Pi)$ so that this limit would effectively give real numbers.

This relates also interestingly to the question whether the notion of p-adic field makes sense for infinite primes. The p-adic norm of any infinite-p p-adic number would be power of π either infinite, zero, or 1. Excluding infinite normed numbers one would have effectively only p-adic integers in the range $1, ...\Pi - 1$ and thus only the Galois field $G < sub > \Pi < /sub >$ representable also as quantum phases.

I conclude with a nice string of text from John'z page:

What's a mathematical phantom? According to Wraith, it's an object that doesn't exist within a given mathematical framework, but nonetheless "obtrudes its effects so convincingly that one is forced to concede a broader notion of existence".

and unashamedly propose that perhaps Galois fields associated with infinite primes might provide this broader notion of existence! In equally unashamed tone I ask whether there exists also hierarchy of conscious entities at $q = 1$ levels in real sense and whether we might identify ourselves as this kind of entities? Note that if cognition corresponds to p-adic space-time sheets, our cognitive bodies have literally infinite geometric size in real sense.

7.7 Jones inclusions in relation to S-matrix and U matrix

TGD leads naturally to zero energy ontology which reduces to the positive energy ontology of the standard model only as a limiting case [C2]. In this framework one must distinguish between the U-matrix characterizing the unitary process associated with the quantum jump (and followed by state function reduction and state preparation) and the S-matrix defining time-like entanglement between positive and negative energy parts of the zero energy state and coding the rates for particle reactions which in TGD framework correspond to quantum measurements reducing time-like entanglement.

7.7.1 S-matrix

In zero energy ontology S-matrix characterizes time like entanglement of zero energy states (this is possible only for HFFs for which $Tr(SS^\dagger) = Tr(Id) = 1$ holds true). S-matrix would code for transition rates measured in particle physics experiments with particle reactions interpreted as quantum measurements reducing time like entanglement. In TGD inspired quantum measurement theory measurement resolution is characterized by Jones inclusion (the group G defines the measured quantum numbers), $\mathcal{N} \subset \mathcal{M}$ takes the role of complex numbers, and state function reduction leads to $\mathcal{N}$ ray in the space $\mathcal{M}/\mathcal{N}$ regarded as $\mathcal{N}$ module and thus from a factor to a sub-factor [C2].

The finite number theoretic braid having Galois group G as its symmetries is the space-time correlate for both the finite measurement resolution and the effective reduction of HFF to that associated with a finite-dimensional quantum Clifford algebra $\mathcal{M}/\mathcal{N}$. $SU(2)$ inclusions would allow angular momentum and color quantum numbers in bosonic degrees of freedom and spin and electro-weak quantum numbers in spinorial degrees of freedom. McKay correspondence would allow to assign to G also compact ADE type Lie group so that also Lie group type quantum numbers could be included in the repertoire.

Galois group G would characterize sub-spaces of the configuration space ("world of classical worlds") number theoretically in a manner analogous to the rough characterization of physical states by using topological quantum numbers. Each braid associated with a given partonic 2-surface would correspond to a particular G that the state would be characterized by a collection of groups G. G would act as symmetries of zero energy states and thus of S-matrix. S-matrix would reduce to a direct integral of S-matrices associated with various collections of Galois groups characterizing the number theoretical properties of partonic 2-surfaces.

It is not difficult to criticize this picture.

1. Why time like entanglement should be always characterized by a unitary S-matrix? Why not some more general matrix? If one allows more general time like entanglement, the description of particle reaction rates in terms of a unitary S-matrix must be replaced with something more general and would require a profound revision of the vision about the relationship between experiment and theory. Also the consistency of the zero energy ontology with positive energy ontology in time scales shorter than the time scale determined by the geometric time interval between positive and negative energy parts of the zero energy state would be lost. Hence the easy way to proceed is to postulate that the universe is self-referential in the sense that quantum states represent the laws of physics by coding S-matrix as entanglement coefficients.

2. Second objection is that there might a huge number of unitary S-matrices so that it would not be possible to speak about quantum laws of physics anymore. This need not be the case since super-conformal symmetries and number theoretic universality pose extremely powerful constraints on S-matrix. A highly attractive additional assumption is that S-matrix is universal in the sense that it is invariant under the inclusion sequences defined by Galois groups G associated with partonic 2-surfaces. Various constraints on S-matrix might actually imply the inclusion invariance.

3. One can of course ask why S-matrix should be invariant under inclusion. One might argue that zero energy states for which time-like entanglement is characterized by S-matrix invariant in the inclusion correspond to asymptotic self-organization patterns for which U-process and state function reduction do not affect the S-matrix in the relabelled basis. The analogy with a fractal asymptotic self-organization pattern is obvious.

7.7.2 U-matrix

In a well-defined sense U process seems to be the reversal of state function reduction. Hence the natural guess is that U-matrix means a quantum transition in which a factor becomes a sub-factor whereas state function reduction would lead from a factor to a sub-factor.

The arguments of [C2] suggest that U matrix could be almost trivial and has as a basic building block the so called factorizing S-matrices for integrable quantum field theories in 2-dimensional Minkowski space. For these S-matrices particle scattering would mean only a permutation of momenta in momentum space. If S-matrix is invariant under inclusion then U matrix should be in a well-defined sense almost trivial apart from a dispersion in zero modes leading to a superpositions of states characterized by different collections of Galois groups.

7.7.3 Relation to TGD inspired theory of consciousness

U-matrix could be almost trivial with respect to the transitions which are diagonal with respect to the number field. What would however make U highly interesting is that it would predict the rates for the transitions representing a transformation of intention to action identified as a p-adic-to-real transition. In this context almost triviality would translate to a precise correlation between intention and action.

The general vision about the dynamics of quantum jumps suggests that the extension of a sub-factor to a factor is followed by a reduction to a sub-factor which is not necessarily the same. Breathing would be an excellent metaphor for the process. Breathing is also a metaphor for consciousness and life. Perhaps the essence of living systems distinguishing them from sub-systems with a fixed state space could be cyclic breathing like process $\mathcal{N} \rightarrow \mathcal{M} \supset \mathcal{N} \rightarrow \mathcal{N}_1 \subset \mathcal{M} \rightarrow$ .. extending and reducing the state space of the sub-system by entanglement followed by de-entanglement.

One could even ask whether the unique role of breathing exercise in meditation practices relates directly to this basic dynamics of living systems and whether the effect of these practices is to increase the value of $\mathcal{M} : \mathcal{N}$ and thus the order of Galois group G describing the algebraic complexity of "partonic" 2-surfaces involved (they can have arbitrarily large sizes). The basic hypothesis of TGD inspired theory of cognition indeed is that cognitive evolution corresponds to the growth of the dimension of the algebraic extension of p-adic numbers involved.

If one is willing to consider generalizations of the existing picture about quantum jump, one can imagine that unitary process can occur arbitrary number of times before it is followed by state function reduction. Unitary process and state function reduction could compete in this kind of situation.

7.7.4 Fractality of S-matrix and translational invariance in the lattice defined by sub-factors

Fractality realized as the invariance of the S-matrix in Jones inclusion means that the S-matrices of $\mathcal{N}$ and $\mathcal{M}$ relate by the projection $P : \mathcal{M} \rightarrow \mathcal{N}$ as $S_{\mathcal{N}} = PS_{\mathcal{M}}P$. $S_{\mathcal{N}}$ should be equivalent with $S_{\mathcal{M}}$ with a trivial re-labelling of strands of infinite braid.

Inclusion invariance would mean translational invariance of the S-matrix with respect to the index n labelling strands of braid defined by the projectors e_i. Translations would act only as a semigroup and S-matrix elements would depend on the difference $m - n$ only. Transitions can occur only for $m - n \geq 0$, that is to the direction of increasing label of strand. The group G leaving $\mathcal{N}$ element-wise invariant would define the analog of a unit cell in lattice like condensed matter systems so that translational invariance would be obtained only for translations $m \rightarrow m + nk$, where one has $n \geq 0$ and k is the number of $M(2, C)$ factors defining the unit cell. As a matter fact, this picture might apply also to ordinary condensed matter systems.

7.8 Category theoretic formulation for quantum measurement theory with finite measurement resolution?

I have been trying to understand whether category theory might provide some deeper understanding about quantum TGD, not just as a powerful organizer of fuzzy thoughts but also as a tool providing genuine physical insights. Marni Dee Sheppeard (or Kea in her blog Arcadian Functor at http://kea-monad.blogspot.com/) is also interested in categories but in much more technical sense. Her dream is to find a category theoretical formulation of M-theory as something, which is not the 11-D something making me rather unhappy as a physicist with second foot still deep in the muds of low energy phenomenology.

7.8.1 Locales, frames, Sierpinski topologies and Sierpinski space

The ideas below popped up when Kea mentioned in M-theory lesson 51 the notions of locale and frame [66]. In Wikipedia I learned that complete Heyting algebras, which are fundamental to category theory, are objects of three categories with differing arrows. CHey, Loc and its opposite category Frm (arrows reversed). Complete Heyting algebras are partially ordered sets which are complete lattices. Besides the basic logical operations there is also algebra multiplication (I have considered the possible role of categories and Heyting algebras in TGD in [E7]). From Wikipedia I also learned that locales and the dual notion of frames form the foundation of pointless topology [67]. These topologies are important in topos theory which does not assume axiom of choice.

The so called particular point topology [68] assumes a selection of single point but I have the physicist's feeling that it is otherwise rather near to pointless topology. Sierpinski topology [69] is this kind of topology. Sierpinski topology is defined in a simple manner: the set is open only if it contains a given preferred point p. The dual of this topology defined in the obvious sense exists also. Sierpinski space consisting of just two points 0 and 1 is the universal building block of these topologies in the sense that a map of an arbitrary space to Sierpinski space provides it with Sierpinski topology as the induced topology. In category theoretical terms Sierpinski space is the initial object in the category of frames and terminal object in the dual category of locales. This category theoretic reductionism looks highly attractive.

7.8.2 Particular point topologies, their generalization, and number theoretical braids

Pointless, or rather particular point topologies might be very interesting from physicist's point of view. After all, every classical physical measurement has a finite space-time resolution. In TGD framework discretization by number theoretic braids replaces partonic 2-surface with a discrete set consisting of algebraic points in some extension of rationals: this brings in mind something which might be called a topology with a set of particular algebraic points. Could this preferred set belongs to any open set in the particular point topology appropriate in this situation?

Perhaps the physical variant for the axiom of choice could be restricted so that only sets of algebraic points in some extension of rationals can be chosen freely and the choices is defined by the intersection of p-adic and real partonic 2-surfaces and in the framework of TGD inspired theory of consciousness would thus involve the interaction of cognition and intentionality with the material world. The extension would depend on the position of the physical system in the algebraic evolutionary hierarchy defining also a cognitive hierarchy. Certainly this would fit very nicely to the formulation of quantum TGD unifying real and p-adic physics by gluing real and p-adic number fields to single super-structure via common algebraic points.

7.8.3 Analogs of particular point topologies at the level of state space: finite measurement resolution

There is also a finite measurement resolution in Hilbert space sense not taken into account in the standard quantum measurement theory based on factors of type I. In TGD framework one indeed introduces quantum measurement theory with a finite measurement resolution so that complex rays become included hyper-finite factors of type II_1 (HFFs).

1. Could topology with particular algebraic points have a generalization allowing a category theoretic formulation of the quantum measurement theory without states identified as complex rays?

2. How to achieve this? In the transition of ordinary Boolean logic to quantum logic in the old fashioned sense (von Neuman again!) the set of subsets is replaced with the set of subspaces of Hilbert space. Perhaps this transition has a counterpart as a transition from Sierpinski topology to a structure in which sub-spaces of Hilbert space are quantum subspaces with complex rays replaced with the orbits of subalgebra defining the measurement resolution. Sierpinski space {0,1} would in this generalization be replaced with the quantum counterpart of the space of 2-spinors. Perhaps one should also introduce q-category theory with Heyting algebra being replaced with q-quantum logic.

7.8.4 Fuzzy quantum logic as counterpart for Sierpinksi space

The program formulated above might indeed make sense. The lucky association induced by Kea's blog was to the ideas about fuzzy quantum logic realized in terms of quantum 2-spinor that I had developed a couple of years ago. Fuzzy quantum logic would reflect the finite measurement resolution. I just list the pieces of the argument.

Spinors and qbits: Spinors define a quantal variant of Boolean statements, qbits. One can however go further and define the notion of quantum qbit, qqbit. I indeed did this for couple of years ago (the last section of this chapter).

Q-spinors and qqbits: For q-spinors the two components a and b are not commuting numbers but non-Hermitian operators: $ab = qba$, q a root of unity. This means that one cannot measure both a and b simultaneously, only either of them. $aa^\dagger$ and $bb^\dagger$ however commute so that probabilities for bits 1 and 0 can be measured simultaneously. State function reduction is not possible to a state in which a or b gives zero. The interpretation is that one has q-logic is inherently fuzzy: there are no absolute truths or falsehoods. One can actually predict the spectrum of eigenvalues of probabilities for say 1. Obviously quantum spinors would be state space counterparts of Sierpinski space and for $q \neq 1$ the choice of preferred spinor component is very natural. Perhaps this fuzzy quantum logic replaces the logic defined by the Heyting algebra.

Q-locale: Could one think of generalizing the notion of locale to quantum locale by using the idea that sets are replaced by sub-spaces of Hilbert space in the conventional quantum logic. Q-openness would be defined by identifying quantum spinors as the initial object, q-Sierpinski space. a (resp. b for the dual category) would define q-open set in this space. Q-open sets for other quantum spaces would be defined as inverse images of a (resp. b) for morphisms to this space. Only for q=1 one could have the q-counterpart of rather uninteresting topology in which all sets are open and every map is continuous.

Q-locale and HFFs: The q-Sierpinski character of q-spinors would conform with the very special role of Clifford algebra in the theory of HFFs, in particular, the special role of Jones inclusions to which one can assign spinor representations of $SU(2)$. The Clifford algebra and spinors of the world of classical worlds identifiable as Fock space of quark and lepton spinors is the fundamental example in which 2-spinors and corresponding Clifford algebra serves as basic building brick although tensor powers of any matrix algebra provides a representation of HFF.

Q-measurement theory: Finite measurement resolution (q-quantum measurement theory) means that complex rays are replaced by sub-algebra rays. This would force the Jones inclusions associated with $SU(2)$ spinor representation and would be characterized by quantum phase q and bring in the q-topology and q-spinors. Fuzzyness of qqbits of course correlates with the finite measurement resolution.

Q-n-logos: For other q-representations of SU(2) and for representations of compact groups (Appendix) one would obtain something which might have something to do with quantum n-logos, quantum generalization of n-valued logic. All of these would be however less fundamental and induced by q-morphisms to the fundamental representation in terms of spinors of the world of classical worlds. What would be however very nice that if these q-morphisms are constructible explicitly it would become possible to build up q-representations of various groups using the fundamental physical realization - and as I have conjectured [C3] - McKay correspondence and huge variety of its generalizations would emerge in this manner.

The analogs of Sierpinski spaces: The discrete subgroups of $SU(2)$, and quite generally, the groups Z_n associated with Jones inclusions and leaving the choice of quantization axes invariant, bring in mind the n-point analogs of Sierpinski space with unit element defining the particular point. Note however that $n \geq 3$ holds true always so that one does not obtain Sierpinski space itself. If all these n preferred points belong to any open set it would not be possible to decompose this preferred set to two subsets belonging to disjoint open sets. Recall that the generalized imbedding space related to the quantization of Planck constant is obtained by gluing together coverings $M^4 \times CP_2 \rightarrow M^4 \times CP_2/G_a \times G_b$ along their common points of base spaces. The topology in question would mean that if some point in the covering belongs to an open set, all of them do so. The interpretation would be that the points of fiber form a single inseparable quantal unit.

Number theoretical braids identified as as subsets of the intersection of real and p-adic variants of algebraic partonic 2-surface define a second candidate for the generalized Sierpinski space with a set of preferred points.

8 Appendix

8.0.5 About inclusions of hyper-finite factors of type II_1

Many names have been assigned to inclusions: Jones, Wenzl, Ocneacnu, Pimsner-Popa, Wasserman [64]. It would seem to me that the notion Jones inclusion includes them all so that various names would correspond to different concrete realizations of the inclusions conjugate under outer automorphisms.

1. According to [64] for inclusions with $\mathcal{M} : \mathcal{N} \leq 4$ (with $A_1^{(1)}$ excluded) there exists a countable infinity of sub-factors with are pairwise non inner conjugate but conjugate to $\mathcal{N}$.

2. Also for any finite group G and its outer action there exists uncountably many sub-factors which are pairwise non inner conjugate but conjugate to the fixed point algebra of G [64]. For any amenable group G the the inclusion is also unique apart from outer automorphism [63].

Thus it seems that not only Jones inclusions but also more general inclusions are unique apart from outer automorphism.

Any *-endomorphism σ, which is unit preserving, faithful, and weakly continuous, defines a sub-factor of type II_1 factor [64]. The construction of Jones leads to a atandard inclusion sequence $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}^1 \subset ....$ This sequence means addition of projectors e_i, $i < 0$, having visualization as an addition of braid strand in braid picture. This hierarchy exists for all factors of type II.

At the limit $\mathcal{M}^\infty = \cup_i \mathcal{M}^i$ the braid sequence extends from $-\infty$ to ∞. Inclusion hierarchy can be understood as a hierarchy of Connes tensor powers $\mathcal{M} \otimes_\mathcal{N} \mathcal{M} .... \otimes_\mathcal{N} \mathcal{M}$. Also the ordinary tensor powers of hyper-finite factors of type II_1 (HFF) as well as their tensor products with finite-dimensional matrix algebras are isomorphic to the original HFF so that these objects share the magic of fractals.

Under certain assumptions the hierarchy can be continued also in opposite direction. For a finite index an infinite inclusion hierarchy of factors results with the same value of index. σ is said to be basic if it can be extended to *-endomorphisms from $\mathcal{M}^1$ to $\mathcal{M}$. This means that the hierarchy of inclusions can be continued in the opposite direction: this means elimination of strands in the braid picture. For finite factors (as opposed to hyper-finite ones) there are no basic *-endomorphisms of $\mathcal{M}$ having fixed point algebra of non-abelian G as a sub-factor [64].

1. Jones inclusions

For hyper-finite factors of type II_1 Jones inclusions allow basic *-endomorphism. They exist for all values of $\mathcal{M} : \mathcal{N} = r$ with $r \in \{4cos^2(\pi/n) | n \geq 3\} \cap [4, \infty)$ [64]. They are defined for an algebra defined by projectors e_i, $i \geq 1$. All but nearest neighbor projectors commute. $\lambda = 1/r$ appears in the relations for the generators of the algebra given by $e_i e_j e_i = \lambda e_i$, $|i - j| = 1$. $\mathcal{N} \subset \mathcal{M}$ is identified as the double commutator of algebra generated by e_i, $i \geq 2$.

This means that principal graph and its dual are equivalent and the braid defined by projectors can be continued not only to $-\infty$ but that also the dropping of arbitrary number of strands is possible [64]. It would seem that ADE property of the principal graph meaning single root length codes for the duality in the case of $r \leq 4$ inclusions.

Irreducibility holds true for $r < 4$ in the sense that the intersection of $Q' \cap P = P' \cap P = C$. For $r \geq 4$ one has $dim(Q' \cap P) = 2$. The operators commuting with Q contain besides identify operator of Q also the identify operator of P. Q would contain a single finite-dimensional matrix factor less than P in this case. Basic *-endomorphisms with $\sigma(P) = Q$ is $\sigma(e_i) = e_{i+1}$. The difference between genuine symmetries of quantum TGD and symmetries which can be mimicked by TGD could relate to the irreducibility for $r < 4$ and raise these inclusions in a unique position. This difference could partially justify the hypothesis [C9] that only the groups $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2, C) \times SU(3)$ define orbifold coverings of $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm / G_a \times G_b$.

2. Wasserman's inclusion

Wasserman's construction of $r = 4$ factors clarifies the role of the subgroup of $G \subset SU(2)$ for these inclusions. Also now $r = 4$ inclusion is characterized by a discrete subgroup $G \subset SU(2)$ and is given by $(1 \otimes \mathcal{M})^G \subset (M_2(C) \times \mathcal{M})^G$. According to [64] Jones inclusions are irreducible also for $r = 4$. The definition of Wasserman inclusion for $r = 4$ seems however to imply that the identity matrices of both $\mathcal{M}^G$ and $(M(2, C) \otimes \mathcal{M})^G$ commute with $\mathcal{M}^G$ so that the inclusion should be reducible for $r = 4$.

Note that G leaves both the elements of $\mathcal{N}$ and $\mathcal{M}$ invariant whereas $SU(2)$ leaves the elements of $\mathcal{N}$ invariant. $M(2, C)$ is effectively replaced with the orbifold $M(2, C)/G$, with G acting as automoprhisms. The space of these orbits has complex dimension $d = 4$ for finite G.

For $r < 4$ inclusion is defined as $M^G \subset M$. The representation of G as outer automorphism must change step by step in the inclusion sequence $... \subset \mathcal{N} \subset \mathcal{M} \subset ...$ since otherwise G would act trivially as one proceeds in the inclusion sequence. This is true since each step brings in additional finite-dimensional tensor factor in which G acts as automorphisms so that although $\mathcal{M}$ can be invariant under $G_\mathcal{M}$ it is not invariant under $G_\mathcal{N}$.

These two inclusions might accompany each other in TGD based physics. One could consider $r < 4$ inclusion $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ with G acting non-trivially in $\mathcal{M}/\mathcal{N}$ quantum Clifford algebra. $\mathcal{N}$ would decompose by $r = 4$ inclusion to $\mathcal{N}_1 \subset \mathcal{N}$ with $SU(2)$ taking the role of G. $\mathcal{N}/\mathcal{N}_1$ quantum Clifford algebra would transform non-trivially under $SU(2)$ but would be G singlet.

In TGD framework the G-invariance for $SU(2)$ representations means a reduction of S^2 to the orbifold S^2/G. The coverings $H_\pm \to H_\pm/G_a \times G_b$ should relate to these double inclusions and $SU(2)$ inclusion could mean Kac-Moody type gauge symmetry for $\mathcal{N}$. Note that the presence of the factor containing only unit matrix should relate directly to the generator d in the generator set of affine algebra in the McKay construction. The physical interpretation of the fact that almost all ADE type extended diagrams ($D_n^{(1)}$ must have $n \geq 4$) are allowed for $r = 4$ inclusions whereas D_{2n+1} and E_6 are not allowed for $r < 4$, remains open.

8.0.6 Generalization from $SU(2)$ to arbitrary compact group

The inclusions with index $\mathcal{M} : \mathcal{N} < 4$ have one-dimensional relative commutant $\mathcal{N}' \cup \mathcal{M}$. The most obvious conjecture that $\mathcal{M} : \mathcal{N} \geq 4$ corresponds to a non-trivial relative commutant is wrong. The index for Jones inclusion is identifiable as the square of quantum dimension of the fundamental representation of $SU(2)$. This identification generalizes to an arbitrary representation of arbitrary compact Lie group.

In his thesis Wenzl [65] studied the representations of Hecke algebras $H_n(q)$ of type A_n obtained from the defining relations of symmetric group by the replacement $e_i^2 = (q-1)e_i + q$. H_n is isomorphic to complex group algebra of S_n if q is not a root of unity and for $q = 1$ the irreducible representations of $H_n(q)$ reduce trivially to Young's representations of symmetric groups. For primitive roots of unity $q = exp(i2\pi/l)$, $l = 4, 5...$, the representations of $H_n(\infty)$ give rise to inclusions for which index corresponds to a quantum dimension of any irreducible representation of $SU(k)$, $k \geq 2$. For $SU(2)$ also the value $l = 3$ is allowed for spin 1/2 representation.

The inclusions are obtained by dropping the first m generators e_k from $H_\infty(q)$ and taking double commutant of both H_∞ and the resulting algebra. The relative commutant corresponds to $H_m(q)$. By reducing by the minimal projection to relative commutant one obtains an inclusion with a trivial relative commutant. These inclusions are analogous to a discrete states superposed in continuum. Thus the results of Jones generalize from the fundamental representation of $SU(2)$ to all representations of all groups $SU(k)$, and in fact to those of general compact groups as it turns out.

The generalization of the formula for index to square of quantum dimension of an irreducible representation of $SU(k)$ reads as

$$\mathcal{M} : \mathcal{N} = \prod_{1 \leq r < s \leq k} \frac{sin^2\left((\lambda_r - \lambda_s + s - r)\pi/l\right)}{sin^2\left((s-r)n/l\right)} \ . \tag{15}$$

Here λ_r is the number of boxes in the r^{th} row of the Yang diagram with n boxes characterizing the representations and the condition $1 \leq k \leq l - 1$ holds true. Only Young diagrams satisfying the condition $l - k = \lambda_1 - \lambda_{r_{max}}$ are allowed.

The result would allow to restrict the generalization of the imbedding space in such a manner that only cyclic group Z_n appears in the covering of $M^4 \to M^4/G_a$ or $CP_2 \to CP_2/G_b$ factor. Be as it may, it seems that quantum representations of any compact Lie group can be realized using the generalization of the imbedding space. In the case of SU(2) the interpretation of higher-dimensional quantum representations in terms of Connes tensor products of 2-dimensional fundamental representations is highly suggestive.

The groups $SO(3,1) \times SU(3)$ and $SL(2,C) \times U(2)_{ew}$ have a distinguished position both in physics and quantum TGD and the vision about physics as a generalized number theory implies them. Also the general pattern for inclusions selects these groups, and one can say that the condition that all possible statistics are realized is guaranteed by the choice $M^4 \times CP_2$.

1. $n > 2$ for the quantum counterparts of the fundamental representation of $SU(2)$ means that braid statistics for Jones inclusions cannot give the usual fermionic statistics. That Fermi

statistics cannot "emerge" conforms with the role of infinite-D Clifford algebra as a canonical representation of HFF of type II_1. $SO(3,1)$ as isometries of H gives Z_2 statistics via the action on spinors of M^4 and $U(2)$ holonomies for CP_2 realize Z_2 statistics in CP_2 degrees of freedom.

2. $n > 3$ for more general inclusions in turn excludes Z_3 statistics as braid statistics in the general case. $SU(3)$ as isometries induces a non-trivial Z_3 action on quark spinors but trivial action at the imbedding space level so that Z_3 statistics would be in question.

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